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On the Electromagnetic Field due to a Moving Electric Doublet.

By

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The electromagnetic field due to an electric doublet moving uniformly in a direction perpendicular to its axis or in a direction of its axis has already been examined by Prof. T. Mizuno¹ as an application of the Principle of Relativity. Six years ago the writer² derived the generalized Lorentz-Einstein transformation equations and noticed that they would have an important bearing on problems concerning motions whose directions are not confined in one of the coordinate axes. The field due to an electric doublet moving in any direction with a constant velocity is easily obtained by making use of the generalized transformation equations.

As is well known, the electric force $\mathbf{E}$ and the magnetic force $\mathbf{H}$, due to a stationary electric doublet, are given by

$$\mathbf{E} = -\frac{\Phi}{r^3} \mathbf{s}_1 + \frac{3\Phi}{r^5} \mathbf{r}(\mathbf{s}_1 \mathbf{r}), \quad \mathbf{H} = 0, \quad (1)$$

where Φ is the electric moment of the doublet, $\mathbf{s}_1$ the unit vector along its axis, and $\mathbf{r}$ the radius vector of any point drawn from its centre.

If the doublet is in motion with a constant velocity $\mathbf{v}$ with respect to a stationary system, and if we consider a moving system, at whose origin the doublet is situated, the electric and magnetic forces measured in the moving system are given by

$$\mathbf{E}' = -\frac{\Phi'}{r'^3} \mathbf{s}_1' + \frac{3\Phi'}{r'^5} \mathbf{r}'(\mathbf{s}_1' \mathbf{r}'), \quad \mathbf{H}' = 0. \quad (2)$$

¹ Mem. Col. Sci. Eng. Kyoto Imp. Univ., 2, 157 (1909).

² Ibid. 3, 103 (1911).

If the two systems coincide at the time $t=t'=0$, we have

$$\mathbf{r}' = \mathbf{r} + (\gamma - 1) \mathbf{v}_1(\mathbf{v}_1\mathbf{r}) - \gamma \mathbf{v}t \quad (3)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$

and $\mathbf{v}_1$ is the unit vector along $\mathbf{v}$, c being the velocity of light in vacuum.

Multiplying $\mathbf{v}$ scalarly and vectorially to (3) and putting

$$\mathbf{r} - \mathbf{v}t = \mathbf{R}, \quad (4)$$

we get

$$\{(\mathbf{v}\mathbf{r}') = \gamma(\mathbf{v}\mathbf{R}), \quad (5)$$

$$[\mathbf{v}\mathbf{r}'] = [\mathbf{v}\mathbf{r}] = [\mathbf{v}\mathbf{R}]. \quad (6)$$

Consequently (3) may be written

$$\mathbf{r}' = \mathbf{R} + (\gamma - 1)\mathbf{v}_1(\mathbf{v}_1\mathbf{R}), \quad (7)$$

or

$$\mathbf{r}' = \gamma\mathbf{R} + (\gamma - 1)[\mathbf{v}_1[\mathbf{v}_1\mathbf{R}]]. \quad (8)$$

Squaring the last equation we have

$$\mathbf{r}'^2 = \gamma^2\mathbf{R}^2\{1 - \beta^2[\mathbf{v}_1\mathbf{R}_1]^2\}, \quad (9)$$

where $\mathbf{R}_1$ is the unit vector along $\mathbf{R}$.

Similarly putting

$$\mathbf{s} - \mathbf{v}t = \mathbf{S}, \quad (10)$$

we obtain

$$\mathbf{s}' = \mathbf{S} + (\gamma - 1)\mathbf{v}_1(\mathbf{v}_1\mathbf{S}), \quad (11)$$

and consequently

$$\mathbf{r}'(\mathbf{s}') = \{\mathbf{R} + (\gamma - 1)\mathbf{v}_1(\mathbf{v}_1\mathbf{R})\} \left\{ (\mathbf{S}\mathbf{R}) + \left(\frac{\gamma}{c}\right)^2 (\mathbf{v}\mathbf{R})(\mathbf{v}\mathbf{S}) \right\}. \quad (12)$$

By the aid of equations (9), (11) and (12), the equation (2) may be written

$$\begin{aligned} \mathbf{E}' = & \frac{e}{\gamma^5 R^5 (1 - \beta^2 [\mathbf{v}_1\mathbf{R}_1]^2)^{5/2}} [3 \{(\mathbf{S}\mathbf{R}) + \left(\frac{\gamma}{c}\right)^2 (\mathbf{v}\mathbf{R})(\mathbf{v}\mathbf{S})\} \mathbf{R} \\ & - \gamma^2 R^2 (1 - \beta^2 [\mathbf{v}_1\mathbf{R}_1]^2) \mathbf{S} \\ & + 3(\gamma - 1)(\mathbf{v}_1\mathbf{R}) \{(\mathbf{S}\mathbf{R}) + \left(\frac{\gamma}{c}\right)^2 (\mathbf{v}\mathbf{R})(\mathbf{v}\mathbf{S})\} \mathbf{v}_1 \\ & - (\gamma - 1)\gamma^2 R^2 (\mathbf{v}_1\mathbf{S}) \{1 - \beta^2 [\mathbf{v}_1\mathbf{R}_1]^2\} \mathbf{v}_1], \\ \mathbf{H}' = & 0, \end{aligned}$$

because it is well known that the electric charge e is an invariant quantity.

If we use θ, ϕ, φ to express angles between the radius vector $\mathbf{R}$ and the velocity $\mathbf{v}$, the axis of the doublet $\mathbf{S}$ and the velocity $\mathbf{v}$, and the radius vector $\mathbf{R}$ and the axis of the doublet $\mathbf{S}$, respectively, and moreover if we denote by Φ the electric moment of the doublet measured in the stationary system, the above expressions take the form

$$\begin{aligned}\mathbf{E}' &= \frac{\Phi}{r^3 R^3 (1 - \beta^2 \sin^2 \theta)^{5/2}} [3\{(1 - \beta^2) \cos \varphi + \beta^2 \cos \theta \cos \phi\} \mathbf{R}_1 \\ &\quad - (1 - \beta^2 \sin^2 \theta) \mathbf{S}_1 \\ &\quad + (\gamma - 1) \{3(1 - \beta^2) \cos \theta \cos \varphi + 2\beta^2 \cos^2 \theta \cos \phi \\ &\quad - (1 - \beta^2) \cos \phi\} \mathbf{v}_1], \\ \mathbf{H}' &= 0.\end{aligned}$$

Substituting these values in the transformation equations of the electric and magnetic forces, namely

$$\begin{cases} \mathbf{E} = \gamma \left\{ \mathbf{E}' - \frac{1}{c} [\mathbf{v} \mathbf{H}'] \right\} - (\gamma - 1) \mathbf{v}_1 (\mathbf{v}_1 \mathbf{E}'), \\ \mathbf{H} = \gamma \left\{ \mathbf{H}' + \frac{1}{c} [\mathbf{v} \mathbf{E}'] \right\} - (\gamma - 1) \mathbf{v}_1 (\mathbf{v}_1 \mathbf{H}'), \end{cases}$$

we get

$$\left. \begin{aligned}\mathbf{E} &= \frac{\Phi(1 - \beta^2)}{R^3(1 - \beta^2 \sin^2 \theta)^{5/2}} [3\{(1 - \beta^2) \cos \varphi + \beta^2 \cos \theta \cos \phi\} \mathbf{R}_1 \\ &\quad - (1 - \beta^2 \sin^2 \theta) \mathbf{S}_1], \\ \mathbf{H} &= \frac{\Phi\beta(1 - \beta^2)}{R^3(1 - \beta^2 \sin^2 \theta)^{5/2}} [3\{(1 - \beta^2) \cos \varphi + \beta^2 \cos \theta \cos \phi\} \sin \theta \mathbf{R}_1^* \\ &\quad - (1 - \beta^2 \sin^2 \theta) \sin \phi \mathbf{S}_1^*],\end{aligned}\right\} \quad (13)$$

where $\mathbf{R}_1^*$ and $\mathbf{S}_1^*$ are the unit vectors along the directions $[\mathbf{v}_1 \mathbf{R}_1]$ and $[\mathbf{v}_1 \mathbf{S}_1]$ respectively. These are the electric and magnetic forces due to an electric doublet moving uniformly in any direction. It is interesting to note that the electric force $\mathbf{E}$ may be considered as made up of two vectors, one in the direction of the axis of the doublet and the other in the direction radial from the centre of the doublet. The electric force is wholly independent of the vector $\mathbf{v}_1$.

Now let us apply the above results to the following special cases :

(i) The doublet is moving along its axis.

In this case, since $\phi=0$ and $\theta=\varphi$, equations (13) reduce to

$$\begin{cases} \mathbf{E} = \frac{\Phi(1-\beta^2)}{K^3(1-\beta^2\sin^2\theta)^{5/2}} [3 \cos \theta \mathbf{R}_1 - (1-\beta^2\sin^2\theta) \mathbf{S}_1], \\ \mathbf{H} = \frac{\Phi\beta(1-\beta^2)}{K^3(1-\beta^2\sin^2\theta)^{5/2}} [3 \cos \theta \sin \theta \mathbf{R}_1^*]. \end{cases}$$

If we take the axis of the doublet as the z -axis we get the following scalar equations

$$\begin{cases} E_x = \frac{3\Phi(1-\beta^2)XZ}{K^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ E_y = \frac{3\Phi(1-\beta^2)YZ}{K^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ E_z = \frac{3\Phi(1-\beta^2)Z^2}{K^5(1-\beta^2\sin^2\theta)^{5/2}} - \frac{\Phi(1-\beta^2)}{K^3(1-\beta^2\sin^2\theta)^{3/2}}, \end{cases} \quad \begin{cases} H_x = -\frac{3\Phi\beta(1-\beta^2)YZ}{K^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ H_y = \frac{3\Phi\beta(1-\beta^2)XZ}{K^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ H_z = 0, \end{cases}$$

where

$$K^2 = X^2 + Y^2 + Z^2.$$

(ii) The doublet is moving perpendicularly to its axis.

In this case, since $\phi=\pi/2$, we get

$$\begin{cases} \mathbf{E} = \frac{\Phi(1-\beta^2)}{K^3(1-\beta^2\sin^2\theta)^{5/2}} [3(1-\beta^2) \cos \varphi \mathbf{R}_1 - (1-\beta^2\sin^2\theta) \mathbf{S}_1], \\ \mathbf{H} = \frac{\Phi\beta(1-\beta^2)}{K^3(1-\beta^2\sin^2\theta)^{5/2}} [3(1-\beta^2) \cos \varphi \sin \theta \mathbf{R}_1^* - (1-\beta^2\sin^2\theta) \mathbf{S}_1^*]. \end{cases}$$

Hence, if the axis of the doublet be taken as the z -axis as before and the direction of motion as the x -axis, we obtain

$$\begin{cases} E_x = \frac{3\Phi(1-\beta^2)^3XZ}{K^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ E_y = \frac{3\Phi(1-\beta^2)^2YZ}{K^5(1-\beta^2\sin^2\theta)^{5/2}}, \\ E_z = \frac{3\Phi(1-\beta^2)^2Z^2}{K^5(1-\beta^2\sin^2\theta)^{5/2}} - \frac{\Phi(1-\beta^2)}{K^3(1-\beta^2\sin^2\theta)^{3/2}}, \end{cases} \quad \begin{cases} H_x = 0, \\ H_y = -\frac{3\Phi\beta(1-\beta^2)Z^2}{K^5(1-\beta^2\sin^2\theta)^{5/2}} + \frac{\Phi\beta(1-\beta^2)}{K^3(1-\beta^2\sin^2\theta)^{3/2}}, \\ H_z = \frac{3\Phi\beta(1-\beta^2)YZ}{K^5(1-\beta^2\sin^2\theta)^{5/2}}. \end{cases}$$

The Electric Field due to a System of Two Linear Oscillators similar in Form to a Bent Antenna.

By

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As the treatment of the electric field due to a bent antenna is not easy, let us here consider a simple case in which an antenna is made up of two linear oscillators, one of which being vertical and the other horizontal, both being so near situated to take an inverted L shape.

For the sake of simplicity, let us assume that the length of the horizontal part of the antenna is twice that of the vertical and that oscillation currents with the same phase are excited in each part.

Now, as is well known, the horizontal component dY and the vertical component dZ of the electric force due to a Hertzian doublet oscillator at a point distant r from its centre are given by

$$\left\{ \begin{aligned} dY &= -\phi \left\{ \left(\frac{3}{r^3} - \frac{m^2}{r} \right) \sin n \left(t - \frac{r}{v} \right) + \frac{3m}{r^3} \cos n \left(t - \frac{r}{v} \right) \right\} \sin \theta \cos \theta \\ &\dots\dots\dots (1), \\ dZ &= \phi \left[-2 \left\{ \frac{1}{r^3} \sin n \left(t - \frac{r}{v} \right) + \frac{m}{r^2} \cos n \left(t - \frac{r}{v} \right) \right\} \right. \\ &\quad \left. + \left\{ \left(\frac{3}{r^3} - \frac{m^2}{r} \right) \sin n \left(t - \frac{r}{v} \right) + \frac{3m}{r^2} \cos n \left(t - \frac{r}{v} \right) \right\} \sin^2 \theta \right] (2), \end{aligned} \right.$$

where ϕ is the electric moment of the doublet and m and n are so related that $m = \frac{2\pi}{\lambda}$ and $\frac{n}{m} = v$, λ being the wave length of the emitted wave and v the velocity of light in vacuum.

If dz be the length of the doublet and Q denote its maximum charge measured in *E. S. U.*, we have

$$\phi = Qdz.$$

Let us assume that the oscillatory current in the oscillator is given by

$$i = Q \frac{n}{v} \sin nt,$$

then (1) and (2) may be written

$$\left\{ \begin{aligned} dY &= -\frac{v}{n} \frac{\partial^2}{\partial y \partial z} \left(\frac{i_t - \frac{r}{v}}{r} \right) dz & \dots\dots\dots (3), \end{aligned} \right.$$

$$\left\{ \begin{aligned} dZ &= \left(\frac{v}{n} \frac{\partial^2}{\partial y^2} + \frac{v}{n} \frac{1}{p} \frac{\partial}{\partial y} \right) \left(\frac{i_t - \frac{r}{v}}{r} \right) dz & \dots\dots\dots (4). \end{aligned} \right.$$

The fundamental oscillation in a linear oscillator of length $2l$ is given by

$$i = a \cos \left(\frac{\pi z}{2l} \right) \sin \left(\frac{\pi v}{2l} \right),$$

where a is the amplitude of the current at its centre.

Since we may consider a long linear oscillator as the combination of many Hertzian doublet oscillators in series, substituting the above value of i in (3) and (4) we get

$$\left\{ \begin{aligned} dY &= \frac{a}{r} \frac{\pi}{2l} \cos \left(\frac{\pi z}{2l} \right) \sin \frac{\pi}{2l} (vt-r) \sin \theta \cos \theta dz \\ &\quad - \frac{3a}{r^2} \cos \left(\frac{\pi z}{2l} \right) \cos \frac{\pi}{2l} (vt-r) \sin \theta \cos \theta dz \\ &\quad - \frac{3a}{r^3} \frac{2l}{\pi} \cos \left(\frac{\pi z}{2l} \right) \sin \frac{\pi}{2l} (vt-r) \sin \theta \cos \theta dz \dots (5), \end{aligned} \right.$$

$$\left\{ \begin{aligned} dZ &= -\frac{a}{r} \frac{\pi}{2l} \cos \left(\frac{\pi z}{2l} \right) \sin \frac{\pi}{2l} (vt-r) \sin^2 \theta dz \\ &\quad + \frac{a}{r^2} \cos \left(\frac{\pi z}{2l} \right) \left(3 \sin^2 \theta - 2 \right) \cos \frac{\pi}{2l} (vt-r) dz \\ &\quad + \frac{a}{r^3} \frac{2l}{\pi} \left(3 \sin^2 \theta - 2 \right) \cos \left(\frac{\pi z}{2l} \right) \sin \frac{\pi}{2l} (vt-r) dz \dots (6). \end{aligned} \right.$$

These expressions give the y, z components of the electric force due to an elementary portion dz of the long linear oscillator. There-

fore, to find the components due to the whole oscillator, we must integrate the above expressions with respect to z from $-l$ to $+l$.

Let us consider the electric force at a point P at a great distance r_0 from the centre of the oscillator, where we may neglect terms containing the fourth power of its reciprocal.

As shown in Fig. I we have

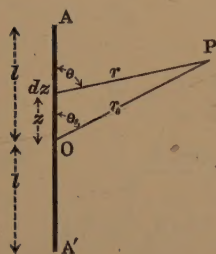


Fig. I.

$$r \cos \theta = r_0 \cos \theta_0 - z \dots\dots\dots (7),$$

$$r \sin \theta = r_0 \sin \theta_0 \dots\dots\dots (8),$$

$$r = \{r_0^2 + z^2 - 2r_0 z \cos \theta_0\}^{\frac{1}{2}}$$

$$= r_0 \left\{ 1 + \frac{z^2}{r_0^2} - 2 \frac{z}{r_0} \cos \theta_0 \right\}^{\frac{1}{2}} \dots\dots\dots (9)',$$

$$= r_0 \left\{ 1 - \frac{z}{r_0} \cos \theta_0 + \frac{1}{2} \frac{z^2}{r_0^2} \sin^2 \theta_0 \right.$$

$$\left. + \frac{1}{2} \frac{z^3}{r_0^3} \sin^2 \theta_0 \cos \theta_0 \right\} \dots\dots\dots (9).$$

By the relations (7) and (8) we get

$$\frac{\sin \theta \cos \theta}{r} = \frac{r_0^2}{r^3} \sin \theta_0 \cos \theta_0 - \frac{r_0}{r^3} z \sin \theta_0$$

$$= \frac{\sin \theta_0 \cos \theta_0}{r_0} \left\{ 1 + \frac{z^2}{r_0^2} - 2 \frac{z}{r_0} \cos \theta_0 \right\}^{-\frac{3}{2}}$$

$$- \frac{z \sin \theta_0}{r_0^2} \left\{ 1 + \frac{z^2}{r_0^2} - 2 \frac{z}{r_0} \cos \theta_0 \right\}^{-\frac{3}{2}} \quad \text{by (9)'}$$

$$= \frac{\sin \theta_0 \cos^2 \theta_0}{r_0} \left\{ 1 - 3 \frac{z}{r_0} \cos \theta_0 + \frac{3}{2} \frac{z^2}{r_0^2} (5 \cos^2 \theta_0 - 1) \right\}$$

$$- \frac{z \sin \theta_0}{r_0^2} \left\{ 1 + 3 \frac{z}{r_0} \cos \theta_0 \right\} \dots\dots\dots (10).$$

Similarly we have

$$\frac{\sin \theta \cos \theta}{r^2} = \frac{\sin \theta_0 \cos \theta_0}{r_0^2} \left\{ 1 + \frac{4z}{r_0} \cos \theta_0 \right\} - \frac{z \sin \theta_0}{r_0^3} \dots\dots (11),$$

$$\frac{\sin \theta \cos \theta}{r^3} = \frac{\sin \theta_0 \cos \theta_0}{r_0^2} \dots\dots\dots (12),$$

$$\frac{\sin^2 \theta}{r} = \frac{\sin^2 \theta_0}{r_0} \left\{ 1 + 3 \frac{z}{r_0} \cos \theta_0 + \frac{3}{2} \frac{z^2}{r_0^2} (5 \cos^2 \theta_0 - 1) \right\}$$

$$\dots\dots\dots (13),$$

$$\frac{\sin^2 \theta}{r^2} = \frac{\sin^2 \theta_0}{r_0^2} \left\{ 1 + 4 \frac{z}{r_0} \cos \theta_0 \right\} \dots\dots\dots (14),$$

$$\frac{\sin^2 \theta}{r_0^3} = \frac{\sin^2 \theta_0}{r_0^3} \dots\dots\dots (15),$$

$$\frac{1}{r^2} = \frac{1}{r_0^2} \left(1 + 2 \frac{z}{r_0} \cos \theta_0 \right) \quad \dots\dots (16),$$

and

$$\frac{1}{r^3} = \frac{1}{r_0^3} \quad \dots\dots (17).$$

Introducing the above several relations into the expressions (5) and (6) we obtain

$$\begin{aligned} dY = & a \left[\frac{\pi}{2l} \cos \left(\frac{\pi z}{2l} \right) \sin \frac{\pi}{2l} \left\{ vt - r_0 + z \cos \theta_0 - \frac{1}{2} \frac{z^2}{r_0} \sin^2 \theta_0 \right. \right. \\ & - \frac{1}{2} \frac{z^3}{r_0^2} \sin^2 \theta_0 \cos \theta_0 \left. \left. \left\{ \frac{\sin \theta_0 \cos \theta_0}{r_0} + \frac{3z \sin \theta_0 \cos^2 \theta_0}{r_0^2} \right. \right. \right. \\ & + \frac{3}{2} \frac{z^2 (5 \cos^2 \theta_0 - 1) \sin \theta_0 \cos \theta_0}{r_0^3} - \frac{z \sin \theta_0}{r_0} - \frac{3z^2 \sin \theta_0 \cos \theta_0}{r_0^3} \left. \left. \right\} \right. \\ & - 3 \cos \left(\frac{\pi z}{2l} \right) \cos \frac{\pi}{2l} \left\{ vt - r_0 + z \cos \theta_0 - \frac{1}{2} \frac{z^2}{r_0} \sin^2 \theta_0 \right. \\ & - \frac{1}{2} \frac{z^3}{r_0^2} \sin^2 \theta_0 \cos \theta_0 \left. \left. \left\{ \frac{\sin \theta_0 \cos \theta_0}{r_0} + 4z \frac{\sin \theta_0 \cos^2 \theta_0}{r_0^3} - \frac{z \sin \theta_0}{r_0^3} \right\} \right. \right. \\ & - 3 \frac{2}{\pi} \cos \left(\frac{\pi z}{2l} \right) \sin \frac{\pi}{2l} \left\{ vt - r_0 + z \cos \theta_0 - \frac{1}{2} \frac{z^2}{r_0} \sin^2 \theta_0 \right. \\ & \left. \left. - \frac{1}{2} \frac{z^3}{r_0^2} \sin^2 \theta_0 \cos \theta_0 \right\} \frac{\sin \theta_0 \cos \theta_0}{r_0^3} \right] dz \quad \dots\dots (18), \end{aligned}$$

$$\begin{aligned} dZ = & a \left[-\frac{\pi}{2l} \cos \left(\frac{\pi z}{2l} \right) \sin \frac{\pi}{2l} \left\{ vt - r_0 + z \cos \theta_0 - \frac{1}{2} \frac{z^2}{r_0} \sin^2 \theta_0 \right. \right. \\ & - \frac{1}{2} \frac{z^3}{r_0^2} \sin^2 \theta_0 \cos \theta_0 \left. \left. \left\{ \frac{\sin^2 \theta_0}{r_0} + 3z \frac{\sin^2 \theta_0 \cos \theta_0}{r_0^2} + \frac{3}{2} \frac{z^2 (5 \cos^2 \theta_0 - 1) \sin^2 \theta_0}{r_0^3} \right\} \right. \right. \\ & + 3 \cos \left(\frac{\pi z}{2l} \right) \cos \frac{\pi}{2l} \left\{ vt - r_0 + z \cos \theta_0 - \frac{1}{2} \frac{z^2}{r_0} \sin^2 \theta_0 \right\} \\ & - \frac{1}{2} \frac{z^3}{r_0^2} \sin^2 \theta_0 \cos \theta_0 \left. \left. \left\{ \frac{\sin^2 \theta_0}{r_0} + 4z \frac{\sin^2 \theta_0 \cos \theta_0}{r_0^3} \right\} \right. \right. \\ & - 2 \cos \left(\frac{\pi z}{2l} \right) \cos \frac{\pi}{2l} \left\{ vt - r_0 + z \cos \theta_0 - \frac{1}{2} \frac{z^2}{r_0} \sin^2 \theta_0 \right. \\ & - \frac{1}{2} \frac{z^3}{r_0^2} \sin^2 \theta_0 \cos \theta_0 \left. \left. \left\{ \frac{1}{r_0^2} + 2z \frac{\cos \theta_0}{r_0^3} \right\} \right. \right. \\ & + 3 \frac{2l}{\pi} \cos \left(\frac{\pi z}{2l} \right) \sin \frac{\pi}{2l} \left\{ vt - r_0 + z \cos \theta_0 - \frac{1}{2} \frac{z^2}{r_0} \sin^2 \theta_0 \right. \\ & \left. \left. - \frac{1}{2} \frac{z^3}{r_0^2} \sin^2 \theta_0 \cos \theta_0 \right\} \frac{\sin^2 \theta_0}{r_0^3} \right] \end{aligned}$$

$$-2 \frac{2l}{\pi} \cos \left(\frac{\pi z}{2l} \right) \sin \frac{\pi}{2l} \left\{ vt - r_0 + z \cos \theta_0 - \frac{1}{2} \frac{z^2}{r_0} \sin^2 \theta_0 - \frac{1}{2} \frac{z^3}{r_0^2} \sin^2 \theta_0 \cos \theta_0 \right\} \frac{1}{r_0^3} \Big] dz \quad \dots (19).$$

Putting

$$\begin{cases} \frac{\pi z}{2l} = u, \\ \sin \frac{\pi}{2l} (vt - r_0) = S, \\ \cos \frac{\pi}{2l} (vt - r_0) = C, \end{cases}$$

and

$$\frac{\pi}{2l} \left(z \cos \theta_0 - \frac{1}{2} \frac{z^2}{r_0} \sin^2 \theta_0 - \frac{1}{2} \frac{z^3}{r_0^2} \sin^2 \theta_0 \cos \theta_0 \right) = A,$$

the equation (19) simplifies to

$$\begin{aligned} dZ &= -adz \frac{\pi}{2l} \cos u \\ &\times \left\{ S \cdot \cos A \left(\frac{\sin^2 \theta_0}{r_0^2} + \frac{3z \sin^2 \theta_0 \cos \theta_0}{r_0^2} + \frac{3}{2} \frac{z^2 (5 \cos^2 \theta_0 - 1) \sin^2 \theta_0}{r_0^3} \right) \right. \\ &+ C \cdot \sin A \left(\frac{\sin^2 \theta_0}{r_0^2} + \frac{3z \sin^2 \theta_0 \cos \theta_0}{r_0^2} + \frac{3}{2} \frac{z^2 (5 \cos^2 \theta_0 - 1) \sin^2 \theta_0}{r_0^3} \right) \Big\} \\ &+ 3adz \cos u \left\{ C \cdot \cos A \left(\frac{\sin^2 \theta_0}{r_0^2} + 4z \frac{\sin^2 \theta_0 \cos \theta_0}{r_0^3} \right) \right. \\ &\quad \left. - S \cdot \sin A \left(\frac{\sin^2 \theta_0}{r_0^2} + 4z \frac{\sin^2 \theta_0 \cos \theta_0}{r_0^3} \right) \right\} \\ &+ 3adz \frac{2l}{\pi} \cos u \left\{ S \cdot \cos A \frac{\sin^2 \theta_0}{r_0^3} + C \cdot \sin A \frac{\sin^2 \theta_0}{r_0^3} \right\} \\ &- 2adz \frac{2l}{\pi} \cos u \left\{ S \cdot \cos A \frac{1}{r_0^3} + C \cdot \sin A \frac{1}{r_0^3} \right\} \\ &= -a \cdot S \left\{ \frac{\pi}{2l} \frac{\sin^2 \theta_0}{r_0} \cos u \cdot \cos A + \frac{\pi}{2l} \frac{3 \sin^2 \theta_0 \cos \theta_0}{r_0^2} z \cos u \cos A \right. \\ &+ \frac{\pi}{2l} \cdot \frac{3}{2} \cdot \frac{(5 \cos^2 \theta_0 - 1) \sin^2 \theta_0}{r_0^3} \cdot z^2 \cdot \cos u \cdot \cos A + 3 \frac{\sin^2 \theta_0}{r_0^2} \cos u \cdot \sin A \\ &+ 3 \cdot 4 \frac{\sin^2 \theta_0 \cos \theta_0}{r_0^3} \cdot z \cdot \cos u \sin A - 2 \frac{1}{r_0^2} \cos u \cdot \sin A \\ &\quad \left. - 2^2 \frac{\cos \theta_0}{r_0^3} z \cos u \sin A - 3 \frac{2l \sin^2 \theta_0}{\pi r_0^3} \cos u \cos A - 2 \frac{2l}{\pi} \frac{1}{r_0^3} \cos u \cos A \right\} dz \\ &- a \cdot C \left\{ \frac{\pi}{2l} \frac{\sin^2 \theta_0}{r_0} \cos u \sin A + \frac{\pi}{2l} \frac{3 \sin^2 \theta_0 \cos \theta_0}{r_0^2} z \cos u \sin A \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{\pi}{2l} \frac{3}{2} \frac{(5 \cos^2 \theta_0 - 1) \sin^2 \theta_0}{r_0^3} z^2 \cos u \sin A - 3 \frac{\sin^2 \theta_0}{r_0^3} \cos u \cos A \\
& - 3 \cdot 4 \frac{\sin^2 \theta_0 \cos \theta_0}{r_0^3} z \cos u \cos A + 2 \frac{1}{r_0^2} \cos u \cos A \\
& + 2^2 \frac{\cos \theta_0}{r_0^3} z \cos u \cos A - 3 \frac{2l}{\pi} \frac{\sin^2 \theta_0}{r_0^3} \cos u \sin A \\
& + 2 \frac{2l}{\pi} \frac{1}{r_0^3} \cos u \sin A \Big\} dz \quad \dots\dots (20).
\end{aligned}$$

Now let us find the integral

$$\begin{aligned}
\int_{-l}^{+l} \frac{\cos u \cos A}{r_0} dz &= \int_{-l}^{+l} \frac{\cos u}{r_0} \left\{ \cos \left(\frac{\pi}{2l} z \cos \theta_0 \right) \cos \left[\frac{\pi}{2l} \frac{z}{2} \sin^2 \theta_0 \left(\frac{z}{r_0} + \frac{z^2}{r_0^2} \cos \theta_0 \right) \right] \right. \\
&\quad \left. + \sin \left(\frac{\pi}{2l} z \cos \theta_0 \right) \sin \left[\frac{\pi}{2l} \frac{z}{2} \sin^2 \theta_0 \left(\frac{z}{r_0} + \frac{z^2}{r_0^2} \cos \theta_0 \right) \right] \right\} dz.
\end{aligned}$$

If we put

$$\frac{\pi}{2l} z \cos \theta_0 = u_0$$

and expand the cosine and sine of the small angle

$$\begin{aligned}
\left[\frac{\pi}{2l} \frac{z}{2} \sin^2 \theta_0 \left(\frac{z}{r_0} + \frac{z^2}{r_0^2} \cos \theta_0 \right) \right] \text{ in series, the integral becomes} \\
\int_{-l}^{+l} \frac{\cos u}{r_0} \left[\cos u_0 \left\{ 1 - \frac{1}{2} \left[\frac{\pi}{2l} \frac{z}{2} \sin^2 \theta_0 \left(\frac{z}{r_0} + \frac{z^2}{r_0^2} \cos \theta_0 \right) \right]^2 \right\} \right. \\
\quad \left. + \sin u_0 \left\{ \frac{\pi}{2l} \frac{z}{2} \sin^2 \theta_0 \left(\frac{z}{r_0} + \frac{z^2}{r_0^2} \cos \theta_0 \right) \right\} \right] dz \\
= \int_{-l}^{+l} \frac{\cos u \cos u_0}{r_0} dz - \frac{1}{2} \frac{\pi^2}{4^2 l^2} \frac{\sin^4 \theta_0}{r_0^3} \int_{-l}^{+l} z^4 \cos u \cos u_0 dz \\
+ \frac{\pi}{4l} \frac{\sin^2 \theta_0}{r_0^2} \int_{-l}^{+l} z^2 \cos u \sin u_0 dz + \frac{\pi}{2l} \frac{\sin^2 \theta_0 \cos \theta_0}{r_0^3} \int_{-l}^{+l} z^3 \cos u \sin u_0 dz.
\end{aligned}$$

But since we have

$$\begin{aligned}
\int_{-l}^{+l} \cos u \cos u_0 dz &= \int_{-l}^{+l} \cos \left(\frac{\pi z}{2l} \right) \cos \left(\frac{\pi}{2l} \cos \theta_0 z \right) dz = \frac{4l}{\pi} \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin^2 \theta_0}, \\
\int_{-l}^{+l} z^4 \cos u \cos u_0 dz &= \frac{4l^5}{\pi \sin^2 \theta_0} \left\{ \cos \left(\frac{\pi}{2} \cos \theta_0 \right) + \frac{4^2 \sin \left(\frac{\pi}{2} \cos \theta_0 \right) \cos \theta_0}{\pi \sin^2 \theta_0} \right\}
\end{aligned}$$

$$\frac{3 \cdot 4^2 \cos\left(\frac{\pi}{2} \cos \theta_0\right)(1 + 3 \cos^2 \theta_0)}{\pi^2 \sin^4 \theta_0} - \frac{3 \cdot 4^2 \sin\left(\frac{\pi}{2} \cos \theta_0\right)(1 + \cos^2 \theta_0) \cos \theta_0}{\pi^3 \sin^6 \theta_0} \\ + \frac{2^7 \cdot 3 \cos\left(\frac{\pi}{2} \cos \theta_0\right)(1 + 10 \cos^2 \theta_0 + 5 \cos^4 \theta_0)}{\pi^4 \sin^8 \theta_0} \left\{ \right. \\ \left. \int_{-l}^{+l} z^2 \cos u \sin u_0 dz = 0 \right.$$

and

$$\int_{-l}^{+l} z^2 \cos u \cos u_0 dz = \frac{4l^4}{\pi \sin^2 \theta_0} \left\{ \sin\left(\frac{\pi}{2} \cos \theta_0\right) - \frac{3 \cdot 4 \cos\left(\frac{\pi}{2} \cos \theta_0\right) \cos \theta_0}{\pi \sin^2 \theta_0} \right. \\ \left. - \frac{2^3 \cdot 3 \sin\left(\frac{\pi}{2} \cos \theta_0\right)(1 + 3 \cos^2 \theta_0)}{\pi^2 \sin^4 \theta_0} + \frac{2^4 \cdot 3 \cdot 4 \cos\left(\frac{\pi}{2} \cos \theta_0\right)(1 + \cos^2 \theta_0) \cos \theta_0}{\pi^3 \sin^6 \theta_0} \right\},$$

the required integral can be written

$$\frac{1}{r_0} \frac{4l}{\pi} \frac{\cos\left(\frac{\pi}{2} \cos \theta_0\right)}{\sin^2 \theta_0} - \frac{\pi^2}{2 \cdot 4^2 l^3} \frac{\sin^4 \theta_0}{r_0^3} \frac{4l^5}{\pi \sin^2 \theta_0} \left\{ \cos\left(\frac{\pi}{2} \cos \theta_0\right) \right. \\ + \frac{4^2 \sin\left(\frac{\pi}{2} \cos \theta_0\right) \cos \theta_0}{\pi \sin^2 \theta_0} - \frac{3 \cdot 4^2 \cos\left(\frac{\pi}{2} \cos \theta_0\right)(1 + 3 \cos^2 \theta_0)}{\pi^2 \sin^4 \theta_0} \\ - \frac{3 \cdot 4^2 \sin\left(\frac{\pi}{2} \cos \theta_0\right)(1 + \cos^2 \theta_0) \cos \theta_0}{\pi^3 \sin^6 \theta_0} \\ + \frac{2 \cdot 3 \cdot 4^2 \cos\left(\frac{\pi}{2} \cos \theta_0\right)(1 + 10 \cos^2 \theta_0 + 5 \cos^4 \theta_0)}{\pi^4 \sin^8 \theta_0} \\ + \frac{\pi}{4l} \frac{\sin^2 \theta_0 \cos \theta_0}{r_0^3} \frac{4l^4}{\pi \sin^2 \theta_0} \left\{ \sin\left(\frac{\pi}{2} \cos \theta_0\right) - \frac{3 \cdot 4 \cos\left(\frac{\pi}{2} \cos \theta_0\right) \cos \theta_0}{\pi \sin^2 \theta_0} \right. \\ \left. - \frac{2^3 \cdot 3 \sin\left(\frac{\pi}{2} \cos \theta_0\right)(1 + 3 \cos^2 \theta_0)}{\pi^2 \sin^4 \theta_0} + \frac{2^4 \cdot 3 \cdot 4 \cos\left(\frac{\pi}{2} \cos \theta_0\right)(1 + \cos^2 \theta_0) \cos \theta_0}{\pi^3 \sin^6 \theta_0} \right\},$$

i.e.

$$\int_{-l}^{+l} \frac{\cos u \cos A}{r_0} dz = \frac{1}{r_0} \frac{4l}{\pi} \frac{\cos\left(\frac{\pi}{2} \cos \theta_0\right)}{\sin^2 \theta_0} - \frac{l^3}{r_0^3} \left\{ \frac{\pi}{2^3} \sin^2 \theta_0 \cos\left(\frac{\pi}{2} \cos \theta_0\right) \right.$$

$$\begin{aligned}
& + \sin\left(\frac{\pi}{2} \cos \theta_0\right) \cos \theta_0 - 2 \cdot 3 \frac{\cos\left(\frac{\pi}{2} \cos \theta_0\right)(1 + \cos^2 \theta_0)}{\pi \sin^2 \theta_0} \\
& + 2 \cdot 3 \frac{\sin\left(\frac{\pi}{2} \cos \theta_0\right) \cos \theta_0 (3 + 11 \cos^2 \theta_0)}{\pi^2 \sin^4 \theta_0} \\
& + 2^4 \cdot 3 \cdot \frac{\cos\left(\frac{\pi}{2} \cos \theta_0\right)(1 + 6 \cos^2 \theta_0 + \cos^4 \theta_0)}{\pi^3 \sin^6 \theta_0} \Bigg\} \dots\dots\dots (21).
\end{aligned}$$

Similarly we get

$$\begin{aligned}
\int_{-1}^{+1} \frac{z \cos u \cos A}{r_0^2} dz &= \frac{l^3}{r_0^3} \left\{ \sin\left(\frac{\pi}{2} \cos \theta_0\right) - 2^2 \cdot 3 \frac{\cos\left(\frac{\pi}{2} \cos \theta_0\right) \cos \theta_0}{\pi \sin^2 \theta_0} \right. \\
& \left. - 2^3 \cdot 3 \frac{\sin\left(\frac{\pi}{2} \cos \theta_0\right)(1 + 3 \cos^2 \theta_0)}{\pi^2 \sin^4 \theta_0} + 2^6 \cdot 3 \frac{\cos\left(\frac{\pi}{2} \cos \theta_0\right)(1 + \cos^2 \theta_0) \cos \theta_0}{\pi^3 \sin^6 \theta_0} \right\} \\
& \dots\dots\dots (22),
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{+1} \frac{z^2 \cos u \cos A}{r_0^3} dz &= \frac{4l^3}{r_0^3} \frac{1}{\pi \sin^2 \theta_0} \left\{ \cos\left(\frac{\pi}{2} \cos \theta_0\right) + 2^3 \frac{\cos \theta_0 \sin\left(\frac{\pi}{2} \cos \theta_0\right)}{\pi \sin^2 \theta_0} \right. \\
& \left. - 2^3 \frac{(1 + 3 \cos^2 \theta_0) \cos\left(\frac{\pi}{2} \cos \theta_0\right)}{\pi^2 \sin^4 \theta_0} \right\} \dots\dots\dots (23),
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{+1} \frac{\cos u \sin A}{r_0} dz &= -\frac{l^2}{r_0^0} \left\{ \cos\left(\frac{\pi}{2} \cos \theta_0\right) + 2^3 \frac{\cos \theta_0 \sin\left(\frac{\pi}{2} \cos \theta_0\right)}{\pi \sin^2 \theta_0} \right. \\
& \left. - 2^3 \frac{(1 + 3 \cos^2 \theta_0) \cos\left(\frac{\pi}{2} \cos \theta_0\right)}{\pi^2 \sin^4 \theta_0} \right\} \dots\dots\dots (24),
\end{aligned}$$

$$\begin{aligned}
\int_{-1}^{+1} \frac{z \cos u \sin A}{r_0^2} dz &= \frac{l^2}{r_0^2} \frac{4}{\pi \sin^2 \theta_0} \left\{ \sin\left(\frac{\pi}{2} \cos \theta_0\right) - 2^2 \frac{\cos \theta_0 \cos\left(\frac{\pi}{2} \cos \theta_0\right)}{\pi \sin^2 \theta_0} \right\} \\
& \dots\dots\dots (25)
\end{aligned}$$

and

$$\int_{-1}^{+1} \frac{z^2 \cos u \sin A}{r_0^3} dz = 0 \dots\dots\dots (26).$$

Using these integrals, we get from (20)

$$\begin{aligned}
 Z &= \int_{-l}^{+l} dZ \\
 &= -aS \left[\frac{\pi}{2l} \sin^2 \theta_0 \left\{ \frac{1}{r_0} \frac{4l}{\pi} \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin^2 \theta_0} - \frac{l^3}{r_0^3} \left(\frac{\pi}{2} \sin^2 \theta_0 \cos \left(\frac{\pi}{2} \cos \theta_0 \right) \right. \right. \right. \\
 &\quad \left. \left. + \sin \left(\frac{\pi}{2} \cos \theta_0 \right) \cos \theta_0 - 2 \cdot 3 \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right) (1 + \cos^2 \theta_0)}{\pi \sin^2 \theta_0} \right. \right. \\
 &\quad \left. \left. + 2 \cdot 3 \frac{\sin \left(\frac{\pi}{2} \cos \theta_0 \right) \cos \theta_0 (3 + 11 \cos^2 \theta_0)}{\pi^2 \sin^4 \theta_0} \right. \right. \\
 &\quad \left. \left. + 2^4 \cdot 3 \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right) (1 + 6 \cos^2 \theta_0 + \cos^4 \theta_0)}{\pi^3 \sin^6 \theta_0} \right\} + \frac{\pi}{2l} \cdot 3 \sin^2 \theta_0 \cos \theta_0 \frac{l^3}{r_0^3} \times \right. \\
 &\quad \times \left\{ \sin \left(\frac{\pi}{2} \cos \theta_0 \right) - 2^2 \cdot 3 \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right) \cos \theta_0}{\pi \sin^2 \theta_0} \right. \\
 &\quad \left. - 2^3 \cdot 3 \frac{\sin \left(\frac{\pi}{2} \cos \theta_0 \right) (1 + 3 \cos^2 \theta_0)}{\pi^2 \sin^4 \theta_0} + 2^6 \cdot 3 \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right) (1 + \cos^2 \theta_0) \cos \theta_0}{\pi^3 \sin^6 \theta_0} \right\} \\
 &\quad + \frac{\pi}{2l} \cdot \frac{3}{2} (5 \cos^2 \theta_0 - 1) \sin^2 \theta_0 \frac{4l^3}{r_0^3} \frac{1}{\pi \sin^2 \theta_0} \left\{ \cos \left(\frac{\pi}{2} \cos \theta_0 \right) \right. \\
 &\quad \left. + 2^3 \frac{\cos \theta_0 \sin \left(\frac{\pi}{2} \cos \theta_0 \right)}{\pi \sin^2 \theta_0} - 2^3 \frac{(1 + 3 \cos^2 \theta_0) \cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\pi^2 \sin^4 \theta_0} \right\} \\
 &\quad - 3 \sin^2 \theta_0 \cdot \frac{l^2}{r_0^3} \left\{ \cos \left(\frac{\pi}{2} \cos \theta_0 \right) + 2^3 \frac{\cos \theta_0 \sin \left(\frac{\pi}{2} \cos \theta_0 \right)}{\pi \sin^2 \theta_0} \right. \\
 &\quad \left. - 2^3 \frac{(1 + 3 \cos^2 \theta_0) \cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\pi^2 \sin^4 \theta_0} \right\} + 3 \cdot 4 \sin^2 \theta_0 \cos \theta_0 \frac{l^2}{r_0^3} \frac{4}{\pi \sin^2 \theta_0} \times \\
 &\quad \times \left\{ \sin \left(\frac{\pi}{2} \cos \theta_0 \right) - 4 \frac{\cos \theta_0 \cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\pi \sin^2 \theta_0} \right\} + 2 \frac{l^2}{r_0^3} \left\{ \cos \left(\frac{\pi}{2} \cos \theta_0 \right) \right. \\
 &\quad \left. + 2^3 \frac{\cos \theta_0 \sin \left(\frac{\pi}{2} \cos \theta_0 \right)}{\pi \sin^2 \theta_0} - 2^3 \frac{(1 + 3 \cos^2 \theta_0) \cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\pi^2 \sin^4 \theta_0} \right\}
 \end{aligned}$$

$$\begin{aligned}
& -2^2 \cos \theta_0 \frac{l^2}{r_0^3} \frac{4}{\pi \sin^2 \theta_0} \left\{ \sin \left(\frac{\pi}{2} \cos \theta_0 \right) - \frac{4}{\pi} \frac{\cos \theta_0 \cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin^2 \theta_0} \right\} \\
& -3 \cdot \frac{2l}{\pi} \sin^2 \theta_0 \frac{1}{r_0^3} \frac{4l}{\pi} \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin^2 \theta_0} + 2 \cdot \frac{2l}{\pi} \frac{1}{r_0^3} \frac{4l}{\pi} \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin^2 \theta_0} \Bigg] \\
& -a \cdot C \left[\frac{\pi}{2l} \sin^2 \theta_0 \left\{ -\frac{l^2}{r_0^2} \left(\cos \left(\frac{\pi}{2} \cos \theta_0 \right) + 2^3 \cos \theta_0 \frac{\sin \left(\frac{\pi}{2} \cos \theta_0 \right)}{\pi \sin^2 \theta_0} \right. \right. \right. \\
& \left. \left. \left. - 2^3 \frac{(1 + 3 \cos^2 \theta_0) \cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\pi \sin^4 \theta_0} \right) \right\} + \frac{\pi}{2l} \cdot 3 \cdot \sin^2 \theta_0 \cos \theta_0 \frac{l^2}{r_0^2} \frac{4}{\pi \sin^2 \theta_0} \times \right. \\
& \times \left\{ \sin \left(\frac{\pi}{2} \cos \theta_0 \right) - 4 \frac{\cos \theta_0 \cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\pi \sin^2 \theta_0} \right\} - 3 \sin^2 \theta_0 \frac{1}{r_0^2} \frac{4l}{\pi} \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin^2 \theta_0} \\
& \left. + 2 \frac{1}{r_0^2} \frac{4l}{\pi} \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin^2 \theta_0} \right].
\end{aligned}$$

This reduces to

$$\begin{aligned}
Z = & -a \sin \frac{\pi}{2l} (vt - r_0) \left[\frac{1}{r_0} \frac{2 \cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin \theta_0} \sin \theta_0 \right. \\
& - \frac{l^2}{r_0^3} \left\{ \frac{\pi^2}{4} \cos \left(\frac{\pi}{2} \cos \theta_0 \right) \sin^3 \theta_0 \sin \theta_0 - \pi \sin \left(\frac{\pi}{2} \cos \theta_0 \right) \sin^2 \theta_0 \cos \theta_0 \right. \\
& - \cos \left(\frac{\pi}{2} \cos \theta_0 \right) (2 - 3 \sin^2 \theta_0) - \frac{45}{\pi} \frac{\sin \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin^3 \theta_0} (1 + \cos^2 \theta_0) \sin \theta_0 \cos \theta_0 \\
& \left. \left. + \frac{48}{\pi^2} \frac{\cos \left(\frac{\pi}{2} \cos \theta_0 \right) \cos^2 \theta_0}{\sin^4 \theta_0} \right\} \right] \\
& + a \cos \frac{\pi}{2l} (vt - r_0) \left[\frac{l}{r_0^2} \left\{ \frac{\pi}{2} \cos \left(\frac{\pi}{2} \cos \theta_0 \right) \sin \theta_0 \sin \theta_0 \right. \right. \\
& \left. \left. - 2 \sin \left(\frac{\pi}{2} \cos \theta_0 \right) \cos \theta_0 \right\} \right] \dots \dots \dots (27).
\end{aligned}$$

Similarly we obtain

$$Y = a \sin \frac{\pi}{2l} (vt - r_0) \left[\frac{1}{r_0} \frac{2 \cos \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin \theta_0} \cos \theta_0 \right.$$

$$\begin{aligned}
 & -\frac{l^3}{r_0^3} \left\{ \frac{\pi^2}{4} \cos\left(\frac{\pi}{2} \cos \theta_0\right) \sin^3 \theta_0 \cos \theta_0 + \frac{\pi}{2} \sin\left(\frac{\pi}{2} \cos \theta_0\right) \sin \theta_0 (1 - 2 \cos^2 \theta_0) \right. \\
 & + \cos\left(\frac{\pi}{2} \cos \theta_0\right) \cos \theta_0 \sin \theta_0 + \frac{1}{\pi} \frac{\sin\left(\frac{\pi}{2} \cos \theta_0\right)}{\sin^3 \theta_0} (61 \cos^2 \theta_0 + 29) \cos^2 \theta_0 \\
 & \quad \left. + \frac{48}{\pi^2} \frac{\cos\left(\frac{\pi}{2} \cos \theta_0\right)}{\sin^3 \theta_0} \cos \theta_0 (1 + 3 \cos^2 \theta_0) \right\} \\
 & - a \cos \frac{\pi}{2l} (vt - r_0) \left[\frac{l}{r_0^2} \left\{ \frac{\pi}{2} \cos\left(\frac{\pi}{2} \cos \theta_0\right) \sin \theta_0 \cos \theta_0 \right. \right. \\
 & \quad \left. \left. + 2 \sin\left(\frac{\pi}{2} \cos \theta_0\right) \sin \theta_0 \right\} \right] \dots\dots\dots (28).
 \end{aligned}$$

These are the expressions for the component electric forces due to a single linear oscillator.

Now in the present problem it is, of course, necessary to take image effects into account; in Fig. II OA is the vertical part of the

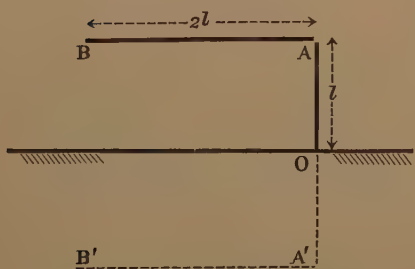


Fig. II.

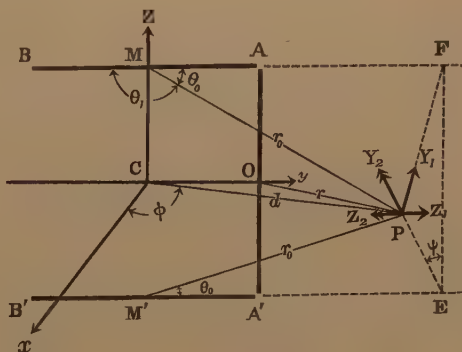


Fig. III.

antenna, AMB its horizontal part, while OA' and $A'M'B'$ are their respective images.

Let us choose the rectangular coordinate system whose origin C is the projection of M , the centre of AB , on the equatorial plane of $A'OA$, the y -axis being taken in the direction of MA and the z -axis in the direction of OA as shown in Fig. III.

Let us find the electric force at a point P , situated on the equatorial plane i.e. on the xy -plane, at a distance d from C , r from O and r_0 from M and its image M' .

Denoting the angles AMP , BMP and XCP by θ_0 , θ_1 and ϕ respectively, we have

$$\left. \begin{aligned} r_0 \cos \theta_0 &= d \sin \phi \\ d^2 + l^2 &= r_0^2 \end{aligned} \right\} \dots\dots\dots (29),$$

$$\begin{aligned} r^2 &= d^2 + l^2 - 2dl \sin \phi \\ &= r_0^2 - 2lr_0 \cos \theta_0, \quad \text{by (29),} \end{aligned}$$

$\therefore$

$$\begin{aligned} r &= r_0 \left\{ 1 - 2 \frac{l}{r_0} \cos \theta_0 \right\}^{\frac{1}{2}} \\ &= r_0 - l \cos \theta_0 - \frac{1}{2} \frac{l^2}{r_0} \cos^2 \theta_0 - \frac{1}{2} \frac{l^3}{r_0^2} \cos^3 \theta_0 \dots\dots\dots (30), \end{aligned}$$

$$\frac{1}{r} = \frac{1}{r_0} \left\{ 1 + \frac{l}{r_0} \cos \theta_0 + \frac{3}{2} \frac{l^2}{r_0^2} \cos^2 \theta_0 \right\} \dots\dots\dots (31),$$

$$\frac{1}{r^2} = \frac{1}{r_0^2} \left\{ 1 + 2 \frac{l}{r_0} \cos \theta_0 \right\} \dots\dots\dots (32),$$

$$\frac{1}{r^3} = \frac{1}{r_0^3} \dots\dots\dots (33).$$

Putting $\theta_0 = \pi/2$ in (27) we get the electric force at P due to the part $A'O A$ in the direction of the z -axis, viz.

$$Z_0 = -a \left\{ \frac{2}{r} - \frac{l^2 \left(1 + \frac{\pi^2}{4} \right)}{r^3} \right\} \sin \frac{\pi}{2l} (vt - r) + a \frac{2l \cdot \frac{\pi}{4}}{r^2} \cos \frac{\pi}{2l} (vt - r).$$

By means of (30)–(33) this may be expressed in terms of r_0 and θ_0 , namely,

$$\begin{aligned} Z_0 &= -a \left[\frac{2}{r_0} \cos \omega + \frac{2l}{r_0^2} \left\{ \frac{\pi}{4} \sin \omega \sin^2 \theta_0 + \cos \omega \cos \theta_0 \right\} \right. \\ &\quad \left. - \frac{l^2}{r_0^3} \left\{ \frac{\pi^2}{4} \cos \omega \sin^4 \theta_0 - \pi \sin \omega \cos \theta_0 \sin^2 \theta_0 + (1 - 3 \cos^2 \theta_0) \cos \omega \right\} \right] \sin \Omega \\ &\quad - a \left[\frac{2}{r_0} \sin \omega - \frac{2l}{r_0^2} \left\{ \frac{\pi}{4} \cos \omega \sin^2 \theta_0 - \sin \omega \cos \theta_0 \right\} \right. \\ &\quad \left. - \frac{l^2}{r_0^3} \left\{ \frac{\pi^2}{4} \sin \omega \sin^4 \theta_0 + \pi \cos \omega \cos \theta_0 \sin^2 \theta_0 + (1 - 3 \cos^2 \theta_0) \sin \omega \right\} \right] \cos \Omega \\ &\dots\dots\dots (34), \end{aligned}$$

where ω and Ω are written for $\frac{\pi}{2} \cos \theta_0$ and $\frac{\pi}{2l} (vt - r_0)$ respectively. But

the horizontal component of the electric force evidently vanishes i.e.

$$Y_0 = 0.$$

Next putting $\theta_0 = \theta_1$ in (27) and (28) we get the electric force due to the part AMB , namely,

$$\begin{aligned} Z_1 &= -a \sin \Omega \left[\frac{2}{r_0} \frac{\cos \left(\frac{\pi}{2} \cos \theta_1 \right)}{\sin \theta_1} \sin \theta_1 - \frac{l^2}{r_0^3} \left\{ \dots \right\} \right] \\ &\quad + a \cos \Omega \frac{2l}{r_0^2} \left\{ \frac{\pi}{4} \cos \left(\frac{\pi}{2} \cos \theta_1 \right) \sin \theta_1 \sin \theta_1 - \sin \left(\frac{\pi}{2} \cos \theta_1 \right) \cos \theta_1 \right\} \\ &= -a \sin \Omega \left[\frac{2}{r_0} \frac{\cos \omega}{\sin \theta_0} \sin \theta_0 - \frac{l^2}{r_0^3} \left\{ \dots \right\} \right] \\ &\quad + a \cos \Omega \frac{2l}{r_0^2} \left\{ \frac{\pi}{4} \cos \omega \sin \theta_0 \sin \theta_0 - \sin \omega \cos \theta_0 \right\} \end{aligned}$$

in the direction of the y -axis, and

$$\begin{aligned} Y_1 &= -a \sin \Omega \left[\frac{2}{r_0} \frac{\cos \omega}{\sin \theta_0} \cos \theta_0 - \frac{l^2}{r_0^3} \left\{ \dots \right\} \right] \\ &\quad + a \cos \Omega \frac{2l}{r_0^2} \left\{ \frac{\pi}{4} \cos \omega \sin \theta_0 \cos \theta_0 + \sin \omega \sin \theta_0 \right\} \end{aligned}$$

in the plane AMP at right angles to AMB .

From (27) and (28) we have at once the electric force due to the part $B'M'A'$, namely

$$\begin{aligned} Z_2 &= -a \sin \Omega \left[\frac{2}{r_0} \frac{\cos \omega}{\sin \theta_0} \sin \theta_0 - \frac{l^2}{r_0^3} \left\{ \dots \right\} \right] \\ &\quad + a \cos \Omega \frac{2l}{r_0^2} \left\{ \frac{\pi}{4} \cos \omega \sin \theta_0 \sin \theta_0 - \sin \omega \cos \theta_0 \right\} \end{aligned}$$

in the direction of the y -axis, and

$$\begin{aligned} Y_2 &= a \sin \Omega \left[\frac{2}{r_0} \frac{\cos \omega}{\sin \theta_0} \cos \theta_0 - \frac{l^2}{r_0^3} \left\{ \dots \right\} \right] \\ &\quad + a \cos \Omega \frac{2l}{r_0^2} \left\{ \frac{\pi}{4} \cos \omega \sin \theta_0 \cos \theta_0 + \sin \omega \sin \theta_0 \right\} \end{aligned}$$

in the plane $A'M'P'$, directed perpendicularly to the part $B'M'A'$.

Since the part $A'M'B'$ is the image of the part AMB , Z_1, Z_2 are equal to one another in magnitude but opposite in direction, while

Y_1, Y_2 are also equal to one another in magnitude, equally inclined by an angle PEF, ϕ say, to, but on the opposite sides of the plane passing through P parallel to the plane of yz , as shown in Fig. III. It is to be noticed that the line PE is drawn perpendicular to the line $M'A'$ and the line EF parallel to the line OA .

Hence, if we assume that the current in the part AB flows from A to B when that in the part OA flows from O to A , the resultant electric force due to AMB and $B'M'A'$ is given by

$$Z_3 = 2 \cdot \cos \phi Y_2.$$

But we have the geometrical relation

$$\cos \phi = \frac{l}{r_0 \sin \theta_0},$$

and therefore we get

$$\begin{aligned} Z_3 &= 2 \frac{l}{r_0 \sin \theta_0} Y_2 \\ &= 2 \cdot \frac{2al}{r_0^2} \frac{\cos \omega}{\sin^2 \theta_0} \cos \theta_0 \sin \Omega - 2 \cdot \frac{2al^2}{r_0^3} \left\{ \frac{\pi}{4} \cos \omega \cos \theta_0 + \sin \omega \right\} \cos \Omega \\ &\dots\dots\dots (35), \end{aligned}$$

where small quantities of the higher orders are neglected.

Finally by adding (34) and (35) we obtain the following electric force at the point P under consideration due to the bent antenna :

$$\begin{aligned} Z &= Z_0 + Z_3 \\ &= -a \left[\frac{2}{r_0} \cos \omega - \frac{2l}{r_0^2} \left\{ \frac{\cos \omega}{\sin^2 \theta_0} (1 + \cos \theta_0) - \frac{\pi}{4} \sin \omega \sin^2 \theta_0 \right\} \right. \\ &\quad - \frac{l^2}{r_0^3} \left\{ \left\{ \left(1 + \frac{\pi^2}{4} \right) - \frac{\pi^2}{4} \cos^2 \theta_0 (1 + \sin^2 \theta_0) - 3 \cos^2 \theta_0 \right\} \cos \omega \right. \right. \\ &\quad \left. \left. - \pi \cos \theta_0 \sin^2 \theta_0 \sin \omega \right\} \right] \sin \Omega \\ &\quad - \left[\frac{2}{r_0} \sin \omega + \frac{2l}{r_0^2} \left\{ \sin \omega \cos \theta_0 - \frac{\pi}{4} \cos \omega \sin^2 \theta_0 \right\} \right. \\ &\quad - \frac{l^2}{r_0^3} \left\{ \left\{ \left(1 + \frac{\pi^2}{4} \right) - \frac{\pi^2}{4} \cos^2 \theta_0 (1 + \sin^2 \theta_0) - 3 \cos^2 \theta_0 \right\} \sin \omega \right. \right. \\ &\quad \left. \left. - \pi \cos^2 \theta_0 \cos \omega - 4 \sin \omega \right\} \right] \cos \Omega. \end{aligned}$$

Denoting the coefficients of $\sin \Omega$ and $\cos \Omega$ by p and q respectively, in the above expression, the amplitude of the electric force is

$$A = \sqrt{p^2 + q^2}$$

$$= \frac{2a}{r_0} \sqrt{1 + 2 \frac{l}{r_0} \cos \theta_0 \left\{ 1 - 2 \frac{\cos^2 \omega}{\sin^2 \theta_0} \right\} + \frac{4l^2}{r_0^2} \left\{ \frac{\cos^2 \omega}{\sin^4 \theta_0} \cos^4 \theta_0 + \cos^2 \theta_0 + \sin^2 \omega - \frac{1}{4} \right\}},$$

where we have neglected small quantities of higher orders.

Therefore the intensity of the electric force at a point on the equatorial plane at a distance r_0 from M or $d = \left(1 - \frac{l^2}{r_0^2}\right)^{\frac{1}{2}}$ from C is proportional to

$$I = 1 + 2 \frac{l}{r_0} \cos \theta_0 \left\{ 1 - 2 \frac{\cos^2 \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin^2 \theta_0} \right\} + \frac{4l^2}{r_0^2} \left\{ \frac{\cos^2 \left(\frac{\pi}{2} \cos \theta_0 \right)}{\sin^4 \theta_0} \cos^4 \theta_0 \right.$$

$$\left. + \cos^2 \theta_0 + \sin^2 \left(\frac{\pi}{2} \cos \theta_0 \right) - 0.25 \right\}.$$

In the positive direction of the y -axis the second term is positive, while in its negative direction this term is negative. Hence the intensity of the electric force at the head-side of the antenna is greater than at its tail side, and the longer the antenna, the greater this directive property.

In order to compare intensities of the electric field at different points lying on a circle of the radius d with its centre at C , it is necessary and sufficient to consider the change of the value I accompanying the change of the angle ϕ .

The relation between ϕ and θ_0 is given by (29) i.e.

$$\sin \phi = \frac{r_0}{d} \cos \theta_0$$

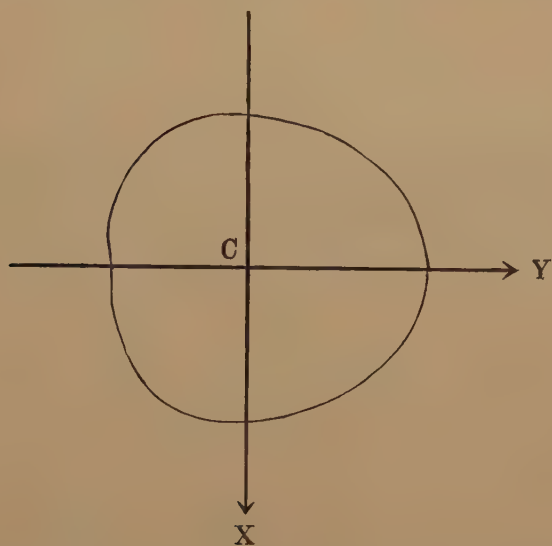
$$= \frac{r_0}{\sqrt{r_0^2 - l^2}} \cos \theta_0$$

$$= \left(1 - \frac{l^2}{r_0^2}\right)^{-\frac{1}{2}} \cos \theta_0.$$

For the sake of convenience, taking the value of d to be $r_0 \left\{ \left(1 - \frac{l^2}{r_0^2}\right)^{\frac{1}{2}} \right\}$ (this means the ratio l/r_0 is equal to 1/10) and using the above relation, the values of I for different values of ϕ are calculated as follows:

θ_0	ϕ	I
5° 44'	90°	1.30
7 37	85	1.30
9 3	83	1.30
10	81 48'	1.30
20	70 41	1.24
30	60 30	1.20
40	50 21	1.12
50	40 15	1.05
60	30 10	0.99
70	20 6	0.96
80	10 3	0.96
85	5 2	0.97
89	1	0.99
89 50	0 10	0.99
90	0	0.99
91	1 10	0.99
95	5 2	1.01
100	10 3	1.03
110	20 6	1.05
120	30 10	1.06
130	40 15	1.03
140	50 21	1.00
150	60 30	0.97
160	70 41	0.94
170	81 48	0.90
170 57	83	0.90
172 23	85	0.90
174 16	90	0.90

In the annexed figure, obtained by plotting the above result, the radius vector of any point on the curve shows the relative intensity of the field in that direction.



The writer wishes to express his sincere thanks to Prof. T. Mizuno and Assist. Prof. K. Tamaki for their suggestions.

Further Investigation on the Figures Produced on Photographic Plates by Electric Discharges.

By

Usaburo Yoshida.

(Received June 15, 1917.)

In the former investigation¹ on the figures produced on photographic plates by electric discharges, beautiful cathode and anode figures were obtained; and a distinct difference between the regular cathode figures and the irregular anode figures was noted. In the anode figures (Fig. 3 and 13 of the former paper) we notice that every branch consists of two parts; namely a more intense portion near the electrode, and a weaker portion more removed. When a celluloid film is used instead of a common photographic plate, these two portions are more clearly distinguishable as is seen in Fig. 18 of the former paper, and Fig. 1 of this paper, the ends of the intense portions of the branches terminating with a continuous outline.

In the former paper the writer tried to explain the formation of the figures, and the transition from brush discharge to spark discharge by the following assumptions: (1) in a cathode branch, the ionisation in its end portion is caused by the collision of negative ions with gas molecules in the air, and these negative ions are supplied by the ionisation due to the collision of positive ions with gas molecules in the neighbourhood of the cathode, or with the cathode itself; (2) in an anode branch also, the ionisation in its end portion is caused by the collision of negative ions with gas molecules, and the ionisation due to the collision of positive ions occurs in the portion nearer to the anode.

When two electrodes placed on a photographic plate, one of which

¹ These Memoirs, 2, 105 (1917).

is connected to the tin foil under the plate, were respectively brought in contact momentarily to both terminals of an influence machine, a discharge figure was produced at the electrode not connected to the tin foil. When this was anode, an anode figure was produced at this electrode.

As already noted in the former paper, though the ends of some anode branches reached the cathode connected to the tin foil, yet no direct spark passed between these two electrodes. This was explained by the two assumptions mentioned above, and also by the consideration that in order to let a direct spark pass between the two electrodes the ionisation due to the collisions of positive ions with the gas molecules or with the cathode itself should occur near the cathode.

When a celluloid film was used instead of a common photographic plate, several photographs were obtained showing that, though the ends of some anode branches of weaker intensity reached the cathode, no direct spark occurred (Fig. 2). In about 60 photographs it was never observed that no direct spark occurred, though the intense portion of the anode figure reached the cathode.

Judging from this we may infer that a direct spark will pass between the electrodes when the end of the intense portion of an anode figure reaches the cathode, but the mere arrival of the weaker portion of an anode figure to the cathode will not cause a direct spark between the electrodes. This may be explained by the supposition that the weaker portion of an anode figure is impressed by the luminosity caused by the ionisation due to the collision of negative ions with the gas molecules, and the ionisation due to the collision of positive ions with gas molecules takes place at the intense portion of the anode figure; because to cause a direct spark to pass between the electrodes the ionisation due to the collision of positive ions should take place near the cathode, as was assumed before.

When a direct spark passed between the electrodes interesting phenomena occurred both at the cathode and anode figures. An enveloping margin of the figure appeared at the cathode leaving a nearly unaffected dark portion between the enveloping margin and the cathode figure as is seen in Fig. 3 of this paper and Fig. 22 of the former paper. Many anode branches which were enveloped by white margins, as if a photographic reversal took place, appeared at the anode as is seen in Fig. 4.

As these phenomena seemed to the writer not to be explainable by mere photographic reversal, a further research was undertaken.

Two electrodes were placed upon a photographic plate laid on a sheet of tin foil, one of these two electrodes being connected to the tin foil. A wire was so attached to one electrode at some distance (2 or 3 cm) above the photographic plate that a direct spark might more easily pass between this wire and the other electrode than on the photographic plate. When these two electrodes were respectively brought in contact momentarily to both terminals of the influence machine, a discharge figure appeared at the electrode which was not connected to the tin foil; and, at the same time, a direct spark passed between the wire and the other electrode. The phenomena before mentioned appeared only in the portion of the photographic plate illuminated by the light of the direct spark; and in the shadow of the electrode cast by the spark, only the ordinary cathode and anode figures were impressed as is seen in Fig. 5 and Fig. 6. This indicates that the light of the direct spark plays an important part in the occurrence of such phenomena.

Even when the intensity of the light emitted by the direct spark was much reduced by making the spark pass at a height of 10 cm or more above the photographic plate, these phenomena were easily observed. And in every case examined the light emitted by the spark was too weak to give rise to an ordinary photographic reversal.

Next, the electrode connected to the tin foil was removed; and the other electrode and the tin foil were so arranged that they might be connected respectively to both terminals of the influence machine. And another spark gap was placed nearly in the position where a direct spark took place in the case of Fig. 5 and Fig. 6. Immediately after both terminals of the influence machine were respectively brought in contact momentarily to the electrode on the photographic plate and the tin foil, they were connected in the circuit of the spark gap above mentioned, and an intense direct spark was seen to have passed at the spark gap. Consequently the illumination of the photographic plate by the light of this direct spark occurred one or two seconds after the impression of the discharge figure on the photographic plate had been made. The cathode figure Fig. 7 and the anode figure Fig. 8 were taken in such a manner.

Though in these cases the illumination of the photographic plate by the light emitted by the spark was comparable with that of the case of Fig. 5 and Fig. 6, no trace of the interesting phenomena before mentioned was observed. When a direct spark was made to

occur one or two seconds before the occurring of the impression of the discharge figure, photographs similar to Fig. 7 and Fig. 8 were obtained, but again no trace of the phenomena before mentioned was detected. These experiments were repeated several times, both on the cathode and anode figure, with various intensities of light emitted by the spark; and the attempt to observe the phenomena before mentioned was entirely unsuccessful.

Consequently we may infer that, in order for the interesting phenomena before mentioned to take place, the illumination of the photographic plate by the light emitted by the spark should occur exactly or nearly at the same time that the discharge figure is impressed on the photographic plate. It is clearly not an ordinary photographic reversal; also it is not a Clayden effect which occurs when a photographic plate exposed to the light of an electric spark or of an intense illumination of short duration, is subsequently fogged by exposure to diffused light before development, because the time interval between the two successive illuminations is not so important a factor in this case; and it does not belong to any kind of photographic reversal ever known.

The fact that, for the formation of the enveloping margin of the discharge figure, the illumination by the light emitted by an electric spark should be applied simultaneously with the formation of the discharge figure on the photographic plate, seems to suggest that the strong electric field applied to the sensitive film has a complex rôle on the sensitiveness of the film to the light in some direct or indirect way.

Metzner¹, by letting many sparks pass on a photographic plate placed on a sheet of tin foil, observed a phenomena similar to that of anode branches, and called such reversed branches "dunkle Funken." An explanation of this phenomenon as a Clayden effect was given by Schaum². As many sparks occurred in the photograph given by Metzner, the appearance of the "dunkle Funken" is not obvious; but, probably, it is the same phenomenon as that observed by the writer. If so, the explanation of the "dunkle Funken" given by Schaum is not correct.

Metzner observed a bright thread in the "dunkle Funken," and this was also observed by the writer on some of the reversed branches

¹ Verh. d. D. phys. Ges., 13, 612 (1911).

² Verh. d. D. phys. Ges., 13, 676 (1911).

in the anode figure impressed on a celluloid film, which was illuminated simultaneously by a spark of a rather weak intensity. But the condition under which the bright thread is formed is not clear. In Fig. 9, such a bright thread appeared on some of the reversed anode branches.

Whether the light, by which the discharge figure was impressed on the sensitive film, was of the same nature as that of the spark or not, was next examined by the Clayden effect; and the result was affirmative. The photographic plate, a discharge figure having been impressed on it, was exposed for a short time to the diffuse light emitted by an electric lamp, and was developed. Both the cathode and the anode figure thus obtained are evidently reversed, as was seen in Fig. 10, and Fig. 11.

When the impression of the figure was preceded by exposure to the diffuse light, no trace of photographic reversal was observed, upon exposures to diffuse light of various intensity or of various duration. Figs. 12 and 13 are examples of photographs taken under such conditions.

Consequently we may conclude that the light which impresses a discharge figure on a sensitive film is of the same nature as that of a spark, so far as the Clayden effect is concerned. R. W. Wood¹ found that the intense light emitted by an electric arc was able to play the part of an electric spark in a Clayden effect, if the interval of time, in which a photographic plate was to be illuminated by it, was reduced below about $1/1000$ sec, and concluded that the peculiarity of the light emitted by a spark with regard to the Clayden effect was no other than its short duration.

As the light which had impressed a discharge figure on a photographic plate showed the Clayden effect, the duration of its illumination in every portion of the figure would be less than $1/1000$ sec. In Fig. 10 and Fig. 11, where the Clayden effect occurred, the reversed cathode and anode branches have no enveloping margins as was seen in Fig. 5 and Fig. 6. This seems to be another evidence in proof that the phenomena observed in Fig. 3 and Fig. 6 are not photographic reversals of any kind ever known.

In conclusion the writer wishes to express his hearty thanks to Prof. T. Mizuno for the interest he has taken in the work.

¹ Phil. Mag., 6, 577 (1903).

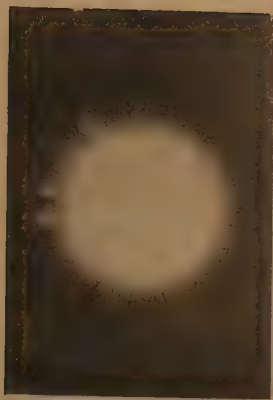


Fig. 1.



Fig. 2.



Fig. 9.

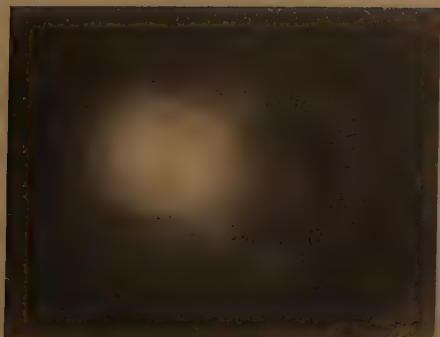


Fig. 3.

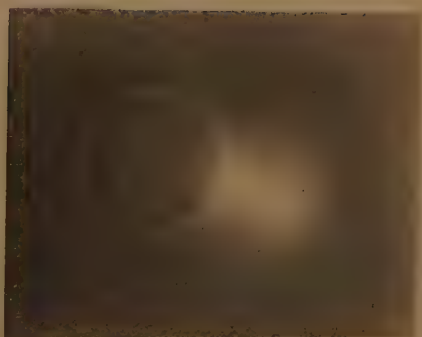


Fig. 4.



Fig. 5.



Fig. 6.



Fig. 7.



Fig. 8.



Fig. 10.



Fig. 11.



Fig. 12.



Fig. 13.

The Effect of an Electric Field on the Spectrum Lines of Hydrogen. Part II.

By

Toshio Takamine and Usaburo Yoshida.

(Received June 15, 1917.)

§ 1. In a previous experiment¹, making use of Lo Surdo's² method, the writers examined the effect of an electric field on the spectrum lines of hydrogen.

In the course of further investigations, relating mainly to the secondary lines of hydrogen, much new data were obtained; and at the same time certain doubtful cases observed in the former experiment were decided.

No essential change was made in experimental method excepting that the fused silica tube was replaced by an ordinary hard glass tube. Experimenting with capillaries having various diameters, we attained an electric field stronger than that in our previous experiment. For photographing the spectrum, the four prism spectrograph mentioned in the former paper was used throughout the experiment.

§ 2. In Fig. 1, a portion of the photograph obtained by this instrument is reproduced; the dispersion at H_{γ} was 14 Å.U. per mm, and in this case the maximum electric field just in front of the cathode was 7.6×10^4 volt/cm.

The various modes presented by the affected secondary lines are shown in Fig. 2. The curves are certainly of an approximate nature and we cannot be sure of the behaviour of those lines in the portion lying above the dotted lines.

Besides the lines above cited, there were many lines which showed

¹ Mem. Coll. Sci., Kyoto, 2, 137 (1917).

² Rendiconti d. Lincei, 22, 664 (1913).

considerable broadening toward the red side in the strong field. These are given in Table I.

TABLE I.

λ in Å.U.	Affected component.
4764.0	p and s
4524.3	s
4511.9	p and s
4490.6	s
4488.0	s
4486.2	s
4461.1	s

Of the lines newly examined, the line λ 4467.3 has already been observed by Stark in the field of 4.8×10^4 volt/cm, but he carried out no precise measurement for this line.

The wide separation of the line λ 4490.6 has already been observed in our first experiment; but, owing to the faint image of the line, it was difficult to measure the amount of the separation at that time.

Including the broadened lines, 15 affected lines were newly found besides the 17 lines reported in the former communication.

In the present experiment, about 60 photographs were taken.

§ 3. In the course of measuring the photographs we noticed that the outer components of the Balmer lines often extended beyond the unaffected neighbouring lines belonging to the secondary spectrum; or, in other words, the branches of the affected Balmer lines crossed through the secondary spectrum lines lying near them.

In the following table we give a list of secondary lines through which the branches of $H\beta$, $H\gamma$, and $H\delta$ crossed.

λ of the secondary spectrum lines.	Balmer lines.
4856.7	p-component of $H\beta$
4335.7	p- and s-components of $H\gamma$
4332.7	" " " " "
4108.1	" " " " $H\delta$
4106.4	" " " " "
4097.6	" " " " "

The problem concerning the origin of the emission of Balmer lines and of the secondary spectrum lines is of great importance. According to Stark,¹ the carriers of the former are the positive atomions, while those of the latter are the neutral atoms of hydrogen. Judging from above cited phenomenon, which seems to show no mutual influence between the secondary lines and the Balmer lines, it seems probable that they are due to different sorts of carriers.

It may be remarked here that, extrapolating the amount of separation of the outer components of H_{γ} (λ 3889 Å.U.) by using the separations in the lower members of Balmer lines, we might expect to see the overlapping of the two opposite branches of H_{ϵ} and H_{γ} at the field of 10^6 volt/cm. To see the existence of mutual influence between the Balmer lines themselves in such a case, may be of no small importance in view of the theory of the atom; but as the intensity of the Balmer lines falls off rapidly with increasing term numbers, there are of course grave experimental difficulties.

Certain regularity in the lines belonging to the secondary spectrum of hydrogen is noted by Fulcher² in his investigation on the spectra of low potential discharges. Fulcher records several groups of lines having constant frequency differences lying in the yellow and red part of the spectrum.

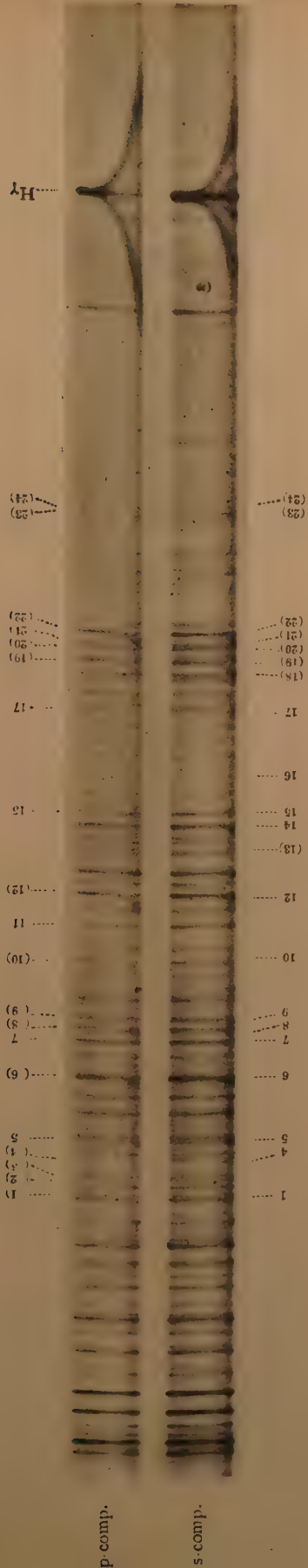
The results obtained in our experiments may be of some use in finding the regularities in the secondary spectrum lines.

In conclusion the writers wish to express their hearty thanks to Prof. T. Mizuno for his kind advice.

¹ Ann. d. Phys., **49**, 179 (1916).

² Phys. ZS., **13**, 1137 (1912).

Fig. 1.



No.	λ in Å.U.	No.	λ in Å.U.
1	4524.3	13	4453.4
2	4520.1	14	4447.2
3	4519.3	15	4445.4
4	4515.3	16	4438.6
5	4511.9	17	4426.1
6	4498.3	18	4419.5
7	4490.6	19	4417.5
8	4488.0	20	4415.3
9	4486.2	21	4414.4
10	4474.4	22	4412.4
11	4467.3	23	4392.1
12	4461.1	24	4391.1

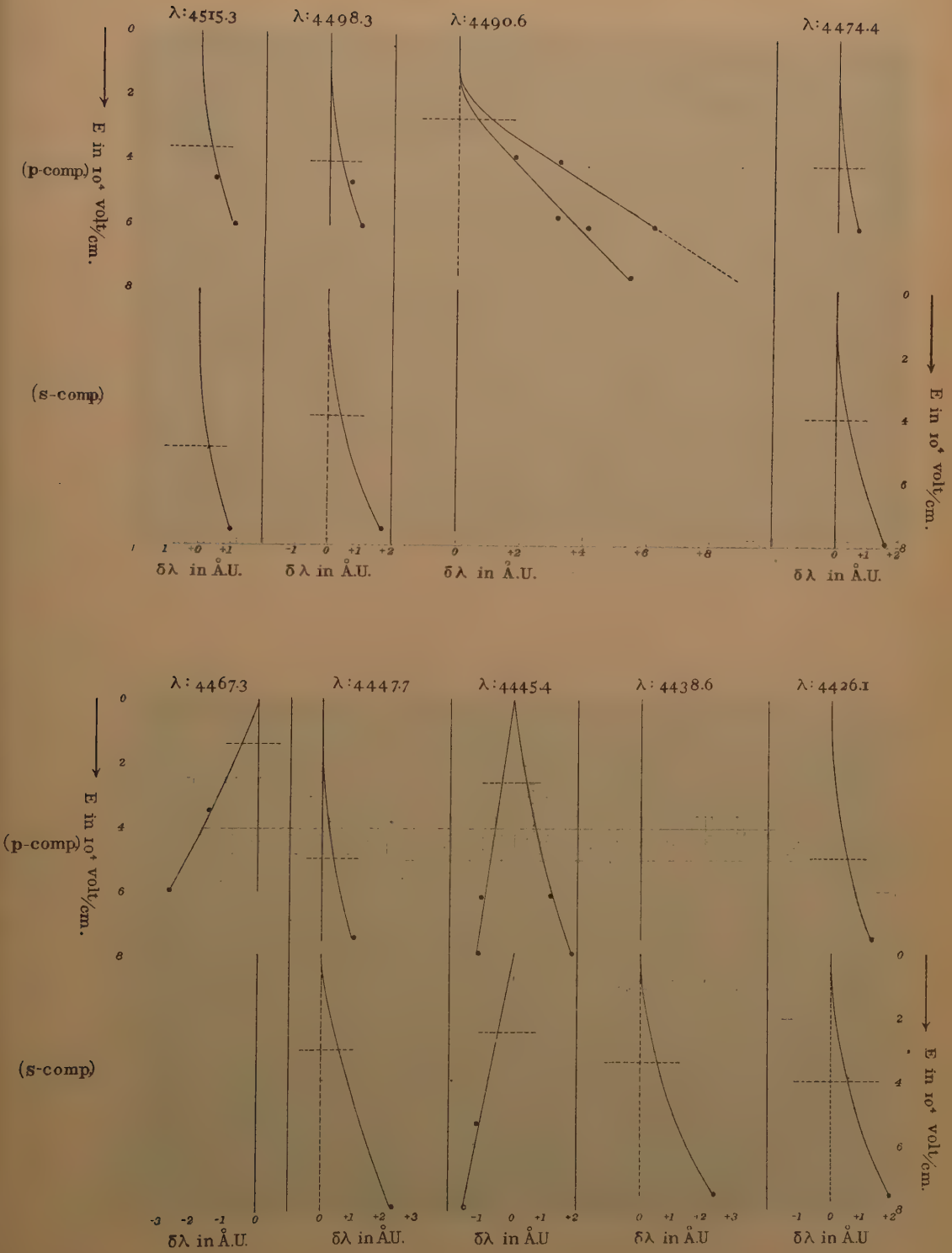
Magnification : 10

Exposure : 3^h 30^m

$E_{\text{max.}}$: 7.6×10^4 volt / cm.

() Observed in our former experiment.

Fig. 2.



The Effect of an Electric Field on the Spectrum Lines of Helium.

By

Toshio Takamine and Usaburo Yoshida.

(Received June 15, 1917.)

§ 1.

In a recently published paper¹, the writers reported an investigation of the Stark effect of hydrogen lines by means of Lo Surdo's² method. With the same experimental arrangements, the research has been extended to the case of helium.

In a series of researches Stark examined the electric effect on a large number of helium lines. Former investigations, of Stark and others, relating to this phenomenon are collected in Stark's monograph³; and it is stated therein that the finer separations of helium lines were observed by Koch. So far as we are aware, however, the detailed account of the investigation by Koch is not yet published, so that we have at present only the data relating to the number of parallel (p) and perpendicular (s) components of the three helium lines λ 4922, 4388, 4026 Å.U. at the field of 10^5 volt/cm.

Lo Surdo's method was first applied to the case of helium by Brunetti. In his first paper⁴, the modes of decomposition of six helium lines 5048, 4922, 4472, 4388, 4144 and 4026 were examined by means

¹ Mem. Coll. Sci., **2**, 137 (1917).

² Rendiconti d. Lincei, **22**, 664 (1913).

³ Elektrische Spectralanalyse chemischer Atome, Leipzig: S. Hirzel, (1914).

⁴ Rendiconti d. Lincei, **23**, 719 (1914).

of a prism spectrograph. In the second experiment¹, Brunetti employed an echelon grating of 12 plates to reveal the behaviour of lines which were very slightly affected by an electric field. In both cases, sketches are given to show the mode of decomposition of different lines; but it is to be regretted that in these figures, p- and s-components are not separately given and further, that no quantitative measurement is attempted.

Lo Surdo's method was also employed by Evans and Croxson² to examine the helium line $\lambda 4686 \text{ \AA.U.}$; but, so far as their preliminary report is concerned, they observed merely the broadening of this line.

Using a field of 10^5 volt/cm. , Koch³ found that a series of new lines $\lambda 6060, 4519, 4046 \text{ \AA.U.}$ appeared in the helium spectrum which could be expressed by a formula of Ritz type. According to Koch, the positions as well as the intensities of these lines depend on the strength of the electric field employed.

§ 2.

The method of the present experiment was the same as described in our former paper, with the exception of a few minor points.

The helium gas was obtained by heating powdered thorinite mineral mixed with double the quantity of KHSO_4 .

The discharge tube was first filled with hydrogen; and, after it was highly evacuated by a Gaede pump, a combustion tube containing the mixture of thorinite and KHSO_4 was slowly heated by a Bunsen burner. The evolved helium gas was passed through two glass tubes containing KOH and P_2O_5 respectively. When the pressure in the discharge tube was reduced until the length of the Crookes dark space became about 2 mm., the side tubes were sealed. The inner diameter of the hard glass capillary tube was in most cases about 1.7 mm., and a current of about 1 milliampere was passed through the secondary circuit.

The cathode disintegration was much heavier than in the case of hydrogen, so that in most cases the exposure was about one hour.

The spectrum obtained was by no means pure; and, beside the hydrogen lines, the band spectrum of CO and certain lines of Al, Hg

¹ Rendiconti. d. Lincei, **24**, 55 (1915).

² Phil. Mag., **32**, 327 (1916).

³ Ann. d. Phys., **48**, 98 (1915).

also made their appearance. The presence of hydrogen lines was, however, convenient for determining the field intensity.

§ 3.

One of the photographs obtained is reproduced in Fig. 1, Pl. I. In the region examined, between λ 5200–4100 Å.U., seven affected lines belonging to the ordinary helium and parhelium spectra were observed.

The following table contains the wave length, series relation, and the dispersion on our photographs for each line.

TABLE I.

λ in Å.U.	Series	Dispersion in Å.U per mm.
5048	Parhelium II. N. 4	41.0
5016	„ H. 3	41.0
4922	„ I. N. 4	40.0
4472	Helium I. N. 4	18.6
4438	Parhelium II. N. 5	16.8
4388	„ I. N. 5	15.0
4121	Helium II. N. 5	8.0

Beside the above lines, the definite separation in the p-component of the line λ 4686 Å.U., and also the appearance of one of the lines found by Koch, at λ 4519 Å.U. were observed.

To determine the intensity of the electric field Stark's data for H_γ line were relied on, as in the case of our former experiment.

In the following we give detailed accounts for each of the lines above mentioned. It is to be remarked that in all the sketches illustrating the mode of separation (Figs. 2, 3, 4, and 5), we cannot be sure of those parts designated by broken lines. The results obtained by Stark are marked on these figures by asterisks.

(1) λ : 4922 (Fig. 2).

The amounts of displacement from the initial line and the roughly estimated intensities of different components at the maximum electric field applied ($E_{\max.}$), are tabulated below.

TABLE II.

$$E_{\text{max.}} = 4.88 \times 10^4 \text{ volt/cm.}$$

p-component.

	v_2	v_1	o (Initial line)	r_1	r_2	r_3
$\delta\lambda$ in Å.U.	-15.70	-4.10	0	+2.26	+5.13	(+7.41 at $E=3.89 \times 10^4 \text{ volt/cm.}$)
Intensity	6	7	2	2	10	(3)

s-component.

	v_2	v_1	o	r_1	r_2
$\delta\lambda$ in Å.U.	-15.50	-4.48	0	+4.60	+8.00
Intensity	5	8	3	10	3

Thus we have 6 p- and 5 s-components, including the faint initial line, while Brunetti gives four components of which three are non-polarised. Of the three points given by Stark for both of the p- and s-components, that lying at -0.8 Å.U. is marked as questionable. In our study, the presence of the faint line at the initial position may have been due to incomplete rectification of the secondary current in the induction coil.

In Stark's monograph it is stated that Koch obtained 7 p- and 7 s-components for the line under consideration, in which two of the latter are doubtful.

As shown in the figure, all the red side components start from the initial line and are displaced proportionally to the intensity of the electric field. On the violet side, one component is, so to speak, entirely isolated, and the other is displaced toward the violet more slowly than is proportional to the field strength.

The isolated component first appears at the field strength of about 15000 volt/cm, as is shown by a small circle in the figure, and gradually gains in its intensity as the field strength is increased.

As will be shown later, the line 4388 is also accompanied by such a component. This phenomenon has already been recorded, in both cases, in one of the papers of Brunetti¹ above cited.

¹ Rendiconti d. Lincei, 23, 719 (1914).

(2) λ : 4472 (Fig. 4).

This line is a member of doublet series; and, according to Runge and Paschen, we have λ 4471.858 (intensity 1) and λ 4471.646 (intensity 6).

TABLE III.

$$E_{max.} = 6.65 \times 10^4 \text{ volt/cm.}$$

p-component.

	v_3	v_2	v_1	0	r_1	r_2	r_3
$\delta\lambda$ in Å.U.	-7.89	-5.70	-3.48	0	+1.42	+4.09	+5.08
Intensity	3	7	1	4	2	10	3

s-component.

	v_3	v_2	v_1	0	r_1
$\delta\lambda$ in Å.U.	-7.50	-5.30	-2.84	0	+3.72
Intensity	3	7	2	5	10

For this line Brunetti records only two components; and Stark gives 3 p- and 3 s-components. In our photograph we could observe 7 p- and 5 s-components, including the initial line; but, owing to over-exposure, the detail is not clearly shown in the reproduction.

In both the p- and the s-components, there is a remarkable feature: the three violet components all start from a point corresponding to about $E = 0.5 \times 10^4$ volt/cm, and $\delta\lambda = -1.3$ Å.U., which is marked by a small circle in Fig. 4.

The disproportionality between the amount of separation and the field strength noted by Stark for this line (Compare Fig. 11 in the monograph of Stark), may be accounted for if we take the above phenomenon in consideration.

According to Paschen and Back¹, the line under consideration is in ordinary state accompanied by a satellite on the violet side. We think, the existence of a satellite must not be overlooked in the present case.

¹ Ann. d. Phys., 39, 897 (1912).

(3) $\lambda: 4388$ (Fig. 5).

For this line Stark found 4 p- and 4 s-components; and Koch, in the case of "Feinzerlegung," observed 7 p- and 6 s-components; while Brunetti noted 5 components of which three were unpolarised. We obtained 6 p- and 5 s-components, including the initial line, as shown in the following table.

TABLE IV.

$$E_{\max.} = 6.65 \times 10^4 \text{ volt/cm.}$$

p-components.

	v_2	v_1	o	r_1	r_2	r_3
$\delta\lambda$ in Å.U.	-16.70	(-4.3 at $E = 2.85 \times 10^4 \text{ volt/cm.}$)	0	+2.74	(+3.65 at $E = 5.53 \times 10^4 \text{ volt/cm.}$)	+11.3
Intensity	8	(4)	6	8	2	10

s-component.

	v_2	v_1	o	r_1	r_2
$\delta\lambda$ in Å.U.	-14.40	-8.20	-0.3	+2.70	+9.20
Intensity	5	4	10	2	6

As may be seen in Fig. 5, the mode of decomposition of this line is similar to that of $\lambda 4922$. The isolated components on the violet side first appear near the point corresponding to $E = 9000 \text{ volt/cm}$ and $\delta\lambda = -4.3 \text{ Å.U.}$

In the s-components the initial line shows a curious behaviour, being displaced about 0.3 Å.U. toward the violet side at the field of $2 \times 10^4 \text{ volt/cm.}$, and thence remaining unaffected in stronger fields.

In connection with this, it is desirable to note that in the experiment of Paschen and Back¹ relating to the Zeeman effect of helium lines, they state as follows:—

"The line $\lambda 4388.100 \text{ Å.U.}$ has no satellite in the natural state, but in the magnetic field, a satellite appears at $\lambda 4387.6 \text{ Å.U.}$ "

Further, it may worthy of note that the violet component lying

¹ Ann. d. Phys., 39, 897 (1912).

nearer to the initial line becomes more and more faint as the field strength is increased, quite contrary to the cases generally observed.

The new appearance of a component at a certain field strength may mean the rise of new electronic vibration. The phenomena were observed for all the three lines $\lambda 4922$, 4472 , and 4388 . The case of 4472 is peculiar in that the new component is split up into three as the field intensity is increased. We believe that these phenomena may have an important bearing on the theory of the atom.

(4) $\lambda 5048$, 5016 , 4438 , 4121 .

All these four lines are only slightly displaced either toward the red or toward the violet by the electric field.

The results of our measurements are tabulated below.

TABLE V.

λ	p-component		s-component	
	$\delta\lambda$ in Å.U.	E in 10^4 volt/cm.	$\delta\lambda$ in Å.U.	E in 10^4 volt/cm.
5048			+1.5	6.65
5016	-1.0	6.65	-0.8	6.65
4438	+2.10	4.88	+2.00	4.88
	+1.30	3.82	+1.04	3.82
	+0.69	2.92	+0.61	2.96
	(+0.80)*	(2.85)	(+0.71)	(2.85)
4121	+0.71	5.70	+0.74	5.70
	+0.51	4.80	+0.52	4.80
	+0.34	4.00	+0.39	4.00

Of the four lines, Stark examined only the line 4438 ; while Brunetti obtained two components for 5048 and for 5016 .

It may be remarked that the effect on the line 4121 , which belongs to the second subordinate series of helium, was perhaps observed for the first time. As in the case of 4472 , the line 4121 is a close doublet; and according to Runge and Paschen we have 4121.143 (intensity 1) and 4120.973 (intensity 3).

The manner in which those helium lines, 4438 and 4121 , are

* Numbers in brackets are those given by Stark.

affected by the electric field is somewhat analogous to that of certain secondary spectrum lines of hydrogen.

(5) $\lambda: 4686$ (Fig. 3), (Dispersion 25.7 A.U. per mm.)

The behavior of this line in an electric field is of special interest. Although the presence of this line in certain celestial spectra has been noted since 1896, it was Fowler¹ who first observed this line in the laboratory. The line has since become of increased importance in connection with Bohr's² theory on the constitution of the atom, according to which this line is attributed to helium. In his later work on enhanced lines, Fowler³ pointed out that this line belongs to the series of enhanced lines whose series constant is four times that of Rydberg (4N type series). This line was observed by Stark⁴ and later by Evans⁵, in the helium tube free from any trace of hydrogen. As was stated before, employing Lo Surdo's method Evans and Croxson⁶ observed the broadening of this line at the field of 37000 volt/cm.

In the present experiment we could clearly observe the separation of the p-component (about 5 Å.U.) and a definite broadening of the s-component at the field of 6.65×10^4 volt/cm.

The portion of Fig. 1 near this line is enlarged in Fig. 6 and the result of measurement is sketched in Fig. 3.

It must be mentioned that the presence of the secondary hydrogen lines $\lambda 4686.94$ and 4690.30 on the red side and $\lambda 4683.96$ on the violet side somewhat blurred the image of the separated components, especially as the intensities of these secondary lines were much increased in the region of the strong field.

(6) $\lambda: 4519$.

Of the new lines found by Koch⁷, we could observe a line $\lambda 4518.8$. Some of the peculiarities noticed by Koch were observed by us also: firstly, both the p- and s-components of this line was displaced about 1 Å.U. toward the red at $E = 6.65 \times 10^4$ volt/cm, and secondly, the p-component was much stronger than the s-component.

¹ Monthly Notices, **73**, 62 (1913).

² Phil. Mag., **26**, 476 (1913).

³ Trans. Roy. Soc., A **214**, 225 (1916).

⁴ Verh. d. Deutsch. phys. Ges., **16**, 468 (1914).

⁵ Phil. Mag., **29**, 284 (1915).

⁶ loc. cit.

⁷ loc. cit.

It may be remarked that for all the lines, 4922, 4472, and 4388, the red side components behave normally, the violet ones anomalously. Further we remark that the separation of the s-components occurs upon a somewhat smaller scale than that of the p-components. As the number of lines investigated in the present experiment is not very large, it would be premature to discuss the behavior of lines with respect to the series relations.

In the former paper the advantage of using a heterogeneous electric field was mentioned. Here it may be emphasized that, in the present experiment, there were many cases where observation in a uniform field only might have proven inadequate.

§ 4. Summary.

1. Employing Lo Surdo's method, the effect of an electric field on the spectrum lines of helium has been investigated.

2. Relying on Stark's data given in the case of "Grobzerlegung," the intensity of the electric field has been calculated from the amount of separation of the outer components of $H\gamma$, which appeared in our photograph together with helium lines.

3. In the fields varying from 3000 to 70000 volt/cm. in strength, the manner of decomposition of the following 7 lines has been examined:—

λ in Å.U. 5048, 5016, 4922, 4472, 4438, 4388, 4121.

The results of our measurements have been compared with those of Stark, Brunetti and Koch.

4. In each of the p- and s-components of the three lines λ 4922, 4472, and 4388 Å.U., an isolated component appeared on the violet side at a certain electric field; and in the case of the 4472 line, the component was further decomposed into three as the field strength was increased.

5. Beside the lines mentioned above, the separation of the p-component of the line 4686 has been observed.

One of the new lines found by Koch has been observed in the present experiment also.

6. Excepting the line 4686, the mode of decomposition of all the helium lines examined has been found to be dissymmetrical with respect to the initial line.

In conclusion, the authors wish to express their hearty thanks to Prof. T. Mizuno for his kind advice; and their great indebtedness to Prof. M. Chikashige of the Chemical Institute for the grant of the thorianite mineral, without which the present experiment could not have been carried out.

Fig. 1

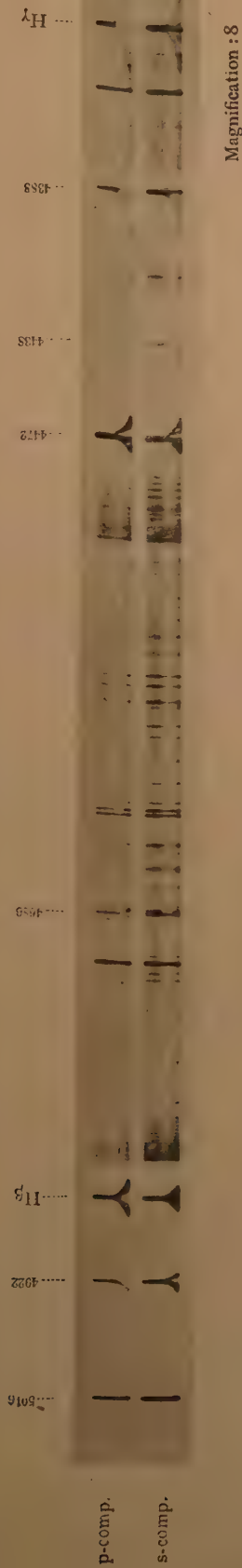
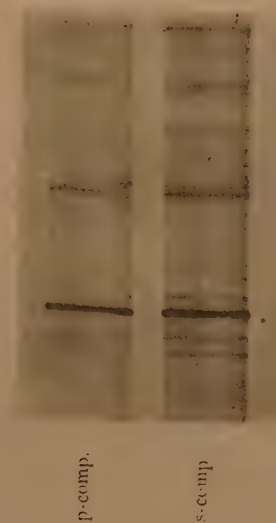


Fig. 6

4024



Exposure : 1 h 15^m

$E_{\text{max.}} : 665 \times 10^4 \text{ volt/cm.}$

Fig. 2. $\lambda: 4922$

$\delta\lambda$ in Å.U.

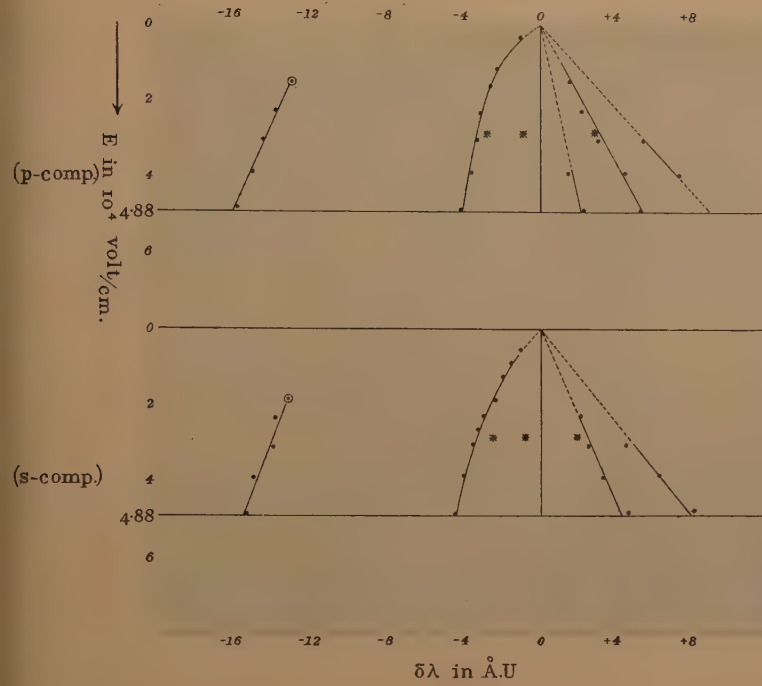


Fig. 3. $\lambda: 4686$

$\delta\lambda$ in Å.U.

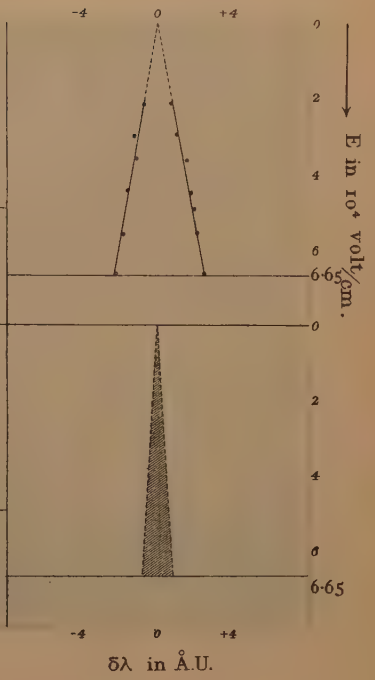


Fig. 4. $\lambda: 4472$

$\delta\lambda$ in Å.U.

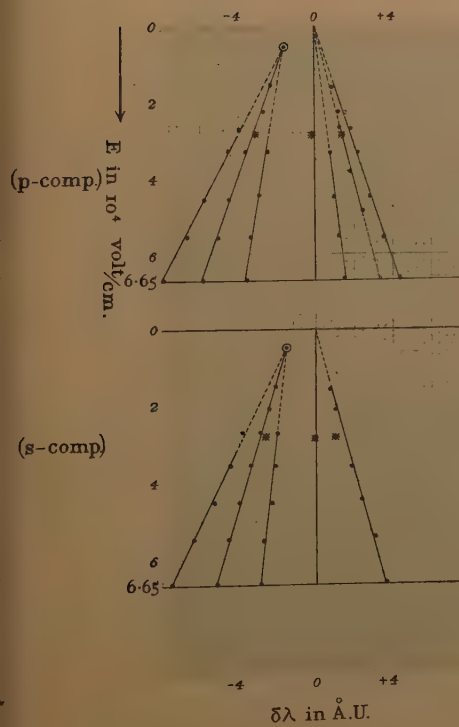
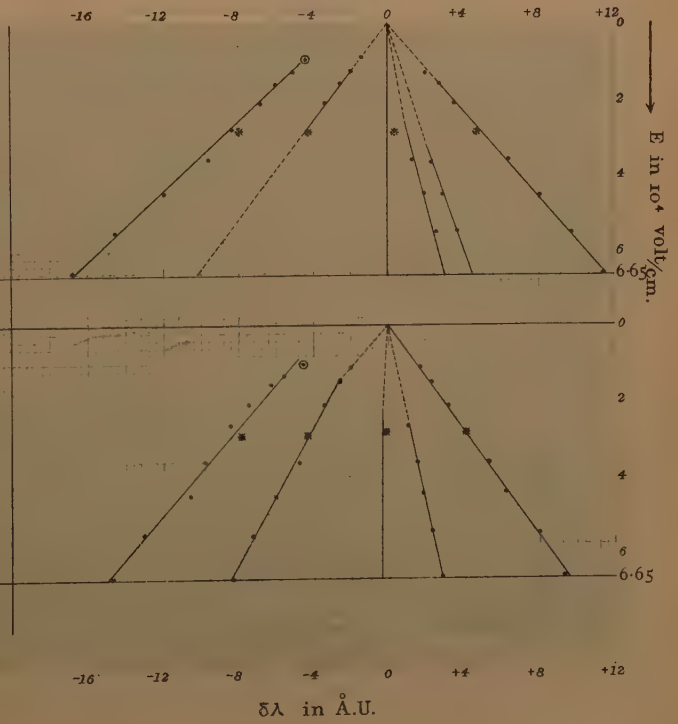


Fig. 5. $\lambda: 4388$

$\delta\lambda$ in Å.U.



Distribution of the Electric Field in the Crookes Dark Space in a Narrow Tube.

By

Toshio Takamine and Usaburo Yoshida.

(Received June 15, 1917.)

According to the investigation by Stark, the relation between the intensity of the electric field and the amount of separation of a certain spectrum line affected by it is well established; and consequently we may determine the distribution of the electric field in the Crookes dark space by utilizing Lo Surdo's¹ method. But it must be remembered that the results obtained by this method are applicable only to a vacuum tube of very small diameter, for with a larger one we encounter many experimental difficulties.

In the present study, the photographs taken in our former investigations² were utilized, some of which have already been published. The gas with which the vacuum tubes were filled was, in most cases, hydrogen, but, in a few cases, it was a mixture of helium and hydrogen.

The glow was never uniform in the dark space, but was mainly concentrated in the axial portion of the tube, excepting the space just in front of the cathode, where a glow of nearly equal intensity occupied the whole portion of the tube. As, in the present experiment, this intense glow in the axial portion was projected on the slit of a spectrograph, it must be remembered that the distribution of the electric field was actually measured in this portion.

For the determination of the electric field, two intense outer (p) and (s) components of the H_{γ} line were taken, as their separation was comparatively great, and the proportionality between the separation and the intensity of the electric field has already been proven by Stark.

¹ Rendiconti d. Lincei, **22**, 664 (1913).

² Mem. Coll. Sci. Kyoto, **2**, 137, 321, 325, (1917).

As the amount of separation of the outer (p) and (s) components, Stark gave the values 4.6 Å.U. and 3.2 Å.U. per 10^4 volt/cm. respectively. These values were employed in the following determination of the intensity of the electric field.

In some cases, the distribution of the electric field in the Crookes dark space was determined by both (p) and (s) components. The curves which represent the relation between the distance of a point in the dark space from the cathode, and the intensity of the electric field at that point were drawn by measuring the (p) and (s) components individually; and a fair agreement of these two curves was seen in every case examined. So in some photographs, where one of these two components were faint, only (p) or (s) components were measured.

With regard to the relation between the potential V of any point in the dark space and its distance x from the cathode, Schuster¹ proposed the following empirical formula.

$$V = V_0(1 - e^{-kx}),$$

where V_0 is the potential in the glow proper, and k a constant, the potential of the cathode being taken as zero. By this formula the intensity of the electric field E at any point in the dark space may be expressed by

$$E = -\frac{dV}{dx} = -V_0 k e^{-kx}.$$

Lo Surdo² observed that the form of the affected spectrum lines of hydrogen due to the glow in the dark space is Y shaped, and concluded that the empirical formula given by Schuster was correct as the first approximation.

Now in our photographs, shown in foregoing papers, the forms of $H\beta$ and $H\gamma$ are not Y shaped, their outer components being curved convex to the central undisplaced lines.

If Schuster's formula be applicable to the present case, the curve which represents the relation between $\log E$ and x should be a straight line. On testing the formula, the result came out negative.

Here it is to be noted that Aston³ and Harris⁴ measured the distribution of the electric field in the Crookes dark space by the deflec-

¹ Proc. Roy. Soc., **47**, 541 (1890).

² Rendiconti d. Lincei, **23**, 117 (1914).

³ Proc. Roy. Soc., **84**, 526 (1910).

⁴ Phil. Mag., **30**, 182 (1915).

tion of cathode rays. But the diameters of the discharge tubes used by them were very large compared with these used in the present experiment.

In a former paper¹ it has already been stated that, if we denote the field intensity at the end of the dark space by E_0 , and that at any point in the dark space by E , the quantity $E-E_0$ seems to increase nearly proportionally to the square of the distance between that point and the end of the dark space. Denoting the distance of any point in the dark space from the cathode by d and the whole length of the dark space by d_0 , the intensity of the electric field E at that point may be represented by the following parabolic relation.

$$E-E_0 = k(d_0-d)^2,$$

where k is a constant. Taking $E_0=0.23$ (expressed in 10^4 volt/cm), $d_0=2.3$ mm., and $k=1.03$, the curve represented in Fig. 3 of the former paper² was drawn. It will be seen that the observed points coincided fairly well with this parabolic curve.

Since then, in the course of our further investigations on the effect of the electric field on the spectrum lines of hydrogen and helium, the formula above mentioned was tested by means of many photographs taken under different experimental conditions; and in every case, it was found that the parabolic formula expressed fairly well the distribution of the electric field in the Crookes dark space.

The three curves shown in Fig. 1 (No. 14, No. 18, and No. 99) were actually drawn by the formula above mentioned, proper constants being chosen for each curve respectively; here the marks $\times$ and $\odot$ denote the observation points. Photographs No. 18³ and No. 99⁴ have already been represented in former papers.

The constants k of the parabolas and various other numerical data for each of the 12 photographs are tabulated in Table I.

¹ Mem. Coll. Sci., Kyoto, 2, 137 (1917).

² Ibid. 2, 137 (1917).

³ Ibid. 2, 325, Fig. 1, (1917).

⁴ Ibid. 2, 321, Fig. 1, (1917).

TABLE I.

I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII
No. of plate	Gas	Current through the tube in milli amp.	Diameter of the tube in mm.	Current density per mm ²	Length of dark space in mm. (observed)	Length of dark space in mm. (calculated)	E_0 in 10^4 volt/cm.	Maximum E in 10^4 volt/cm.	k	$k \times$ (dark space) $\times$ (diameter)	Deviation from the mean value	Cathode fall in volt (Calculated)
14	He and H ₂	0.6	1.72	0.26	1.4	1.59	0.29	4.88	1.76	4.8	-0.4	2820
18	He and H ₂	0.6	1.72	0.26	1.7	1.75	0.41	6.53	2.00	6.0	+0.8	4300
37	H ₂	1.2	2.00	0.38	2.3	2.30	0.23	5.85	1.03	4.8	-0.4	4700
77	H ₂	1.7	2.4	0.38	2.5	2.64	0.28	7.30	0.86	5.5	+0.3	6240
80	H ₂	1.3	2.4	0.29	2.6	2.59	0.35	5.35	0.75	4.7	-0.5	5310
88	H ₂	0.7	1.5	0.40	2.1	1.94	0.26	7.00	1.89	5.5	+0.3	5100
89	H ₂	0.9	1.5	0.51	1.6	1.64	0.35	5.50	1.95	4.8	-0.4	3430
90	H ₂	0.7	1.5	0.44	2.0	2.03	0.30	6.90	1.64	4.9	-0.3	5190
97	H ₂	0.7	1.7	0.31	1.5	1.57	0.40	5.27	1.96	5.5	-0.3	3160
99	H ₂	0.7	1.7	0.31	2.0	2.13	0.29	7.60	1.61	5.8	+0.6	5810
102	H ₂	0.6	1.42	0.38	1.1	1.22	0.35	4.42	2.78	4.8	+0.4	2110
104	H ₂	0.4	1.77	0.16	2.5	2.31	0.25	7.80	1.28	5.2 mean 5.2	0	5830

From the table it will be seen that the intensity E_0 of the electric field at the end of the dark space has a value lying between 2300 volt/cm. and 4100 volt/cm.; but, as the percentage errors are large, further consideration of each value will lead to no definite conclusion.

Although the value of the parabolic constant k varies within a wide range, yet its value, multiplied by the product of the length of the dark space and the diameter of the tube, remains practically constant, as shown in the eleventh column of the table.

The deviation of each value from the mean is, roughly speaking, less than 10% ; and this may be ascribed to experimental errors.

As the diameter of the tube and the length of the dark space are, in the present experiment, restricted within a narrow range, the constancy of the quantity $k \times (\text{dark space}) \times (\text{diameter})$ may have no general meaning. But for practical purposes, it may serve for an estimation of the intensity of the electric field in the Crookes dark space in a narrow tube.

If we take the constant value of the above quantity as 5.2, the distribution of the electric field in the Crookes dark space may be represented, in our study, by the equation

$$E = E_0 + \frac{5.2}{d_0 \times \text{diameter}} (d_0 - d)^2;$$

and, consequently, the maximum electric field E_{max} , just in front of the cathode is given by

$$E_{\text{max}} = E_0 + 5.2 \frac{d_0}{\text{diameter}}.$$

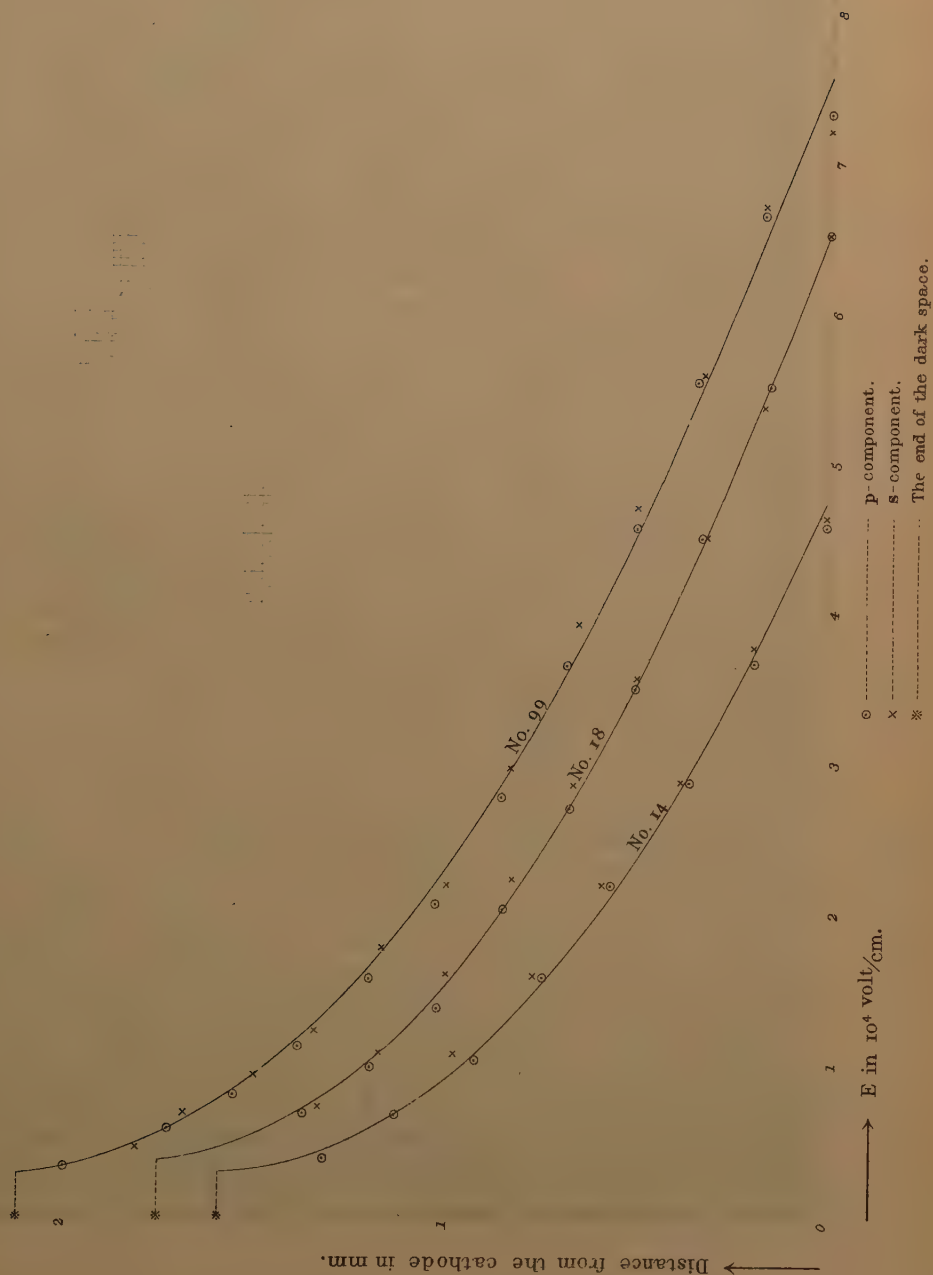
Thus in order to increase the intensity of the electric field just in front of the cathode we have to increase the length of the dark space and to diminish the diameter of the capillary tube. But, with such small tubes as we used, when the length of the dark space was increased, the margin of the dark space became indistinct; and at last a sudden change occurred in the appearance of the dark space. The weak glow which formerly had existed in the dark space disappeared, and a glow intense as that in the positive column occupied that portion of the dark space. In this state of discharge, no trace of the electric separation of the spectrum lines of hydrogen could be observed. This state determines the upper limit of the length of the dark space. For example, with a tube whose diameter was 1.5 mm., this sudden change

occurred when the length of the dark space was increased above 2.5 mm. Though this maximum length of the dark space increased with the diameter of the tube, yet the numerical relation between them was not examined. The maximum electric field we arrived at was 7.8×10^4 volt/cm., the length of the dark space being 2.5 mm, and the diameter of the tube being 1.77 mm.

Lo Surdo states that the length of the dark space is independent of the strength of the current when the pressure is kept constant. But, in our case, a certain increment in the length of the dark space was observed when the current was increased.

In conclusion, the authors' cordial thanks are due to Prof. T. Mizuno for his kind advice.

Fig. 1.



On the Influence of a Magnetic Field and of Mechanical Stirring on the Potential Difference of an Electrolytic Cell.

By

Toshikazu Mashimo.

(Received June 28, 1917.)

In the present experiment the writer made an investigation on the effect of a magnetic field on the potential difference between the electrodes in an electrolytic cell, and also the influence of mechanical stirring on it when a steady current was sent through the cell.

When a steady current is sent through an electrolyte, polarisation occurs at the electrodes; and it seems that the magnetic effect on the electrodes has an intimate relation to it.

Polarisation always takes place in a living tissue when an electric current flows through it. A living body is built up of many cells, each of which consists of protoplasm and a semipermeable membrane; and, when an electric current flows from one cell to another, a difference of concentration of ions, and, consequently, polarisation takes place at the membrane. Further investigation of such polarisation is highly necessary in electrophysiological researches.

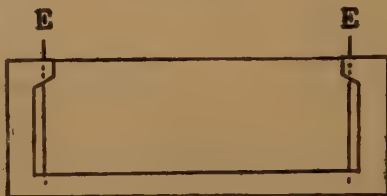
Besides the effect of a magnetic field on the electrodes, the writer examined whether a mechanical stirring of the solution near the electrodes has any influence on the electrical nature of the system, and found that the effect of such stirring of the solution in the neighbourhood of one electrode on the potential difference between the electrodes is similar in nature to that of the magnetic field when it is applied to the same electrode.

Method of Experiment.

The experiments were carried on by the following two methods.

In one method, a Wheatstone bridge was employed, and in the other, the potential difference between the electrodes was measured by a galvanometer connected to the electrodes in parallel with the battery circuit.

The electrolytic cell used was prepared in the following manner. An ebonite plate was cut out in the form indicated in the annexed figure, both sides being closed with glass plates. At both ends of the ebonite frame, platinum or iron wires *E* were supported, and served as the electrodes. Many cells of various sizes were used, the largest



one being $20 \times 3 \times 2$ cm in size. The platinum electrodes were platinized before each experiment; and were well washed with distilled water. The iron electrodes were ordinary iron wires, and they were polished before each experiment. The galvanometer employed was one of a d'Arsonval type. The electromotive force was applied between the electrodes of the cell by means of one or two storage batteries or dry batteries connected in series. The electrolyte used was half mol. iron trichloride solution etc., and the cell was subjected to the magnetic field due to an electromagnet, whose maximum intensity was 6900 Gauss.

In the Wheatstone bridge method, the zero point of the needle of the galvanometer was not always constant in consequence of variation in the internal condition of the cell; but a certain amount of deflection was clearly observed when a magnetic field was applied to the electrodes.

In the other method, the cell, a storage battery and a resistance box was connected in series; and, in parallel with the cell, there was a high resistance circuit, a small portion of which being shunted by the galvanometer. In this case also the zero point of the galvanometer needle was not constant.

The mechanical stirring of the solution in the cell, was performed with the same arrangement described above.

Experiment.

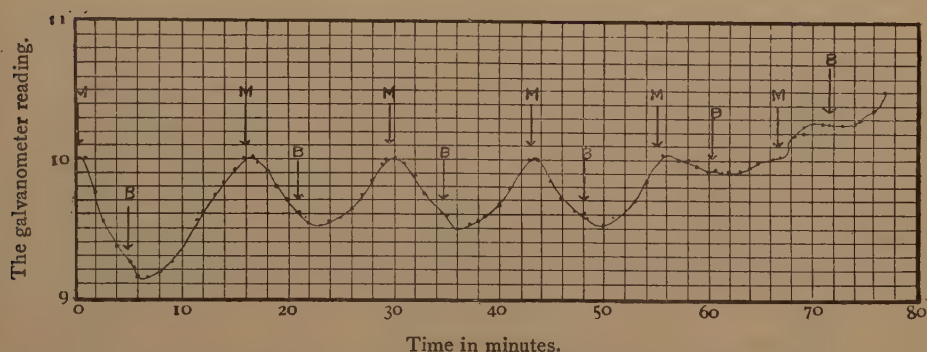
I. The magnetic field applied on both electrodes and its repeated application.

The current in the battery circuit was so adjusted that no gas

bubbles appeared at the electrodes. The electrolyte used was a half mol. iron trichloride solution. The platinum electrodes were platinized, and they were both placed between the poles of the electromagnet.

When the magnetic field was applied, a variation of the deflection of the galvanometer needle occurred similar to that of the diminution of the potential difference due to the polarisation of the cell.

By making successive applications of the same given magnetic field, at the intervals shown in the following curve, the effect became less and less.



M denotes the time at which the magnetic field was applied.

B denotes the time at which the magnetic field was taken off.

II. The relation between the intensity of the magnetic field and its effect.

A magnetic field of various strengths, from 3000 Gauss up to 6900 Gauss, was applied. The arrangement was the same as that of experiment I. The effect increased with the magnetic field, but finally approached to a certain limit asymptotically.

III. Application of the magnetic field on each of the electrodes.

In this case, by using the cell of suitable size, the distance between the electrodes was made sufficiently great so that when one electrode was placed between the poles of the electromagnet the other was far away from them.

The effect was much greater when the cathode was placed in the magnetic field than when the anode was placed in it. When iron electrodes were employed instead of platinum ones, the result was complex as is to be described in V.

IV. Experiment with platinum electrodes in other salts.

When a ferrous sulphate solution or a potassium ferricyanide solution was used, the effect was similar in nature to that in the case of an iron trichloride solution, but of less intensity. When a cobalt nitrate solution was used, the effect was slight; and in the case of nickel chloride solution it was still much slighter.

V. Application of the magnetic field on each of the iron electrodes.

The arrangement was the same as that in experiment III, the iron electrodes being used.

(a). With a sufficiently strong magnetic field the effect was similar to that obtained in the case of the platinum electrodes, but with a weak magnetic field the galvanometer needle showed an increment of the potential difference of the cell when it was applied to the anode.

(b). When a certain relatively strong current was sent through the battery circuit, the application of the magnetic field on either of the electrodes caused a diminution of the potential difference between the electrodes. But when the current was weak, in the case of the anode there was a very slight increment of the potential difference between the electrodes, whilst in the case of the cathode there was the usual diminution.

VI. The electromotive force of the cell set up in consequence of the stirring of the electrolyte.

As electrodes the writer used platinum, iron, copper and nickel. The electrodes, which consisted of a pair of one of those metals, were each time immersed in a proper solution. When the solution in the neighbourhood of one of the electrodes was stirred, a galvanometer deflection was observed, which further increased with violent stirring, but finally reached a certain limiting value.

The result of the experiments is given in the following table.

The metal used as the electrode.	The solution.	Potential at the disturbed electrode.	Potential at the other undisturbed electrode.
iron	half mol. iron trichloride	+	-
nickel	1 mol. nickel chloride	+	-
platinum	dilute hydrochloric acid	-	+
copper	dilute hydrochloric acid	-	+

VII. Mechanical stirring of the solution.

The apparatus was just the same as that in experiment V.

The effect of mechanical stirring of the electrolyte round an electrode was examined. The electrolyte round one electrode was stirred mechanically with a fine glass rod, care being taken as much as possible not to affect the portion of the electrolyte round the other electrode.

The effect was more or less similar in nature to the case of a magnetic field, but more conspicuous. When the current in the battery circuit was weak, mechanical stirring of the electrolyte near the cathode caused a diminution of the potential difference between the electrodes, whilst the stirring near the anode caused a nearly equal amount of increment. When the current in the battery circuit was made stronger, the increment of the potential difference due to the stirring near the anode became smaller.

It was here to be remarked that in all the experiments mentioned above the results obtained were not quantitative but rather qualitative.

Summary.

With platinum electrodes, immersed in a dilute iron trichloride solution, the effect of a magnetic field on the potential difference between the electrodes was especially noticeable when it was applied to the cathode. The diminution of the potential difference between the electrodes was less when a cobalt- or nickel-salt solution was employed.

With the iron electrodes the effect was more complex. When a weak magnetic field was applied to the anode, the potential difference between the electrodes increased, and when it was applied to the cathode, it diminished. But with a strong magnetic field, whether applied to the anode or cathode, a decrement of the potential difference was always observed.

With a relatively strong electric current in the battery circuit, the magnetic field caused the diminution of the potential difference between the electrodes when it was applied to either of the electrodes. But with a weak current, a small increment of the potential difference was observed in the case of the anode.

Mechanical stirring of the electrolyte near one electrode affected the potential difference between the electrodes similarly as in the case of the application of the magnetic field to that electrode. With a weak current in the battery circuit, the potential difference diminished when the electrolyte near the cathode was stirred, and increased when the electrolyte near the anode was stirred. When the current in the bat-

tery circuit was made stronger the effect at the cathode was especially remarkable compared with that at the anode.

A mere stirring of the electrolyte near one electrode gives rise to a new electromotive force between the electrodes of the same metal immersed in its own, or other salt solutions. When the electrolyte in the neighbourhood of the metal was stirred, the latter became anode or cathode, according to the nature of the metal and the solution used.

On the potential difference arising between a magnetized iron and unmagnetized one immersed in an electrolyte, Th. Gross,¹ H. A. Rowland and L. Bell,² G. O. Squier,³ E. L. Nichol and W. S. Franklin,⁴ Hurmuzescu⁵, Bucherer⁶, U. Lala and A. Fournier⁷ and Paillet⁸ have already made investigations.

From the present experiment it will be seen that such potential difference as is caused by the magnetization of one electrode surely has a certain effect on the change of the potential fall between the electrodes. This is however not sufficient to explain experiment V.

Bucherer found that, when a magnetic field was applied to one of the iron electrodes immersed in an electrolyte, a potential difference was set up between them; and a mechanical stirring of the electrolyte round the same electrode had the same effect as that due to a magnetic field. This phenomenon was observed by the writer also.

Procopiu⁹ observed a change of the potential difference between the electrodes caused by a movement of one of the electrodes of the cell. This phenomenon is somewhat analogous to that described in experiment VI, which may at present be explained in a following rather premature way.

At the contact surface of a metal and an electrolyte a potential difference due to the solution pressure of the metal takes place, the metal and the solution being oppositely charged. By stirring the electrolyte in the neighbourhood of the metal, this potential difference will

¹ Sitzungsber. der kais. Ak. der Wiss., **92**, Dec. (1885).

² Jour. of Sc., (3) **36**, 39 (1888).

³ Am. J., **45**.

⁴ Sill. Jour., (3) **35**, 290 (1888).

⁵ C.R., **119**, 1006 (1894).

⁶ Wied. Ann., **58**, 564 (1896).

⁷ C. R., **123**, 801 (1896).

⁸ C. R., **131**, 1194 (1900).

⁹ Bull. d. la Sect. Scintif. d. l'Acad. Roma, **3**, 187 (1915).

be disturbed, and tends to become neutralized; whilst as, at the other electrode, the potential difference due to the solution pressure remains the same, a potential difference will be set up between the electrodes.

If we admit this mode of explanation, experiment VII also seems to be explainable. When the external electromotive force was applied to the electrodes of the cell, the potential difference between the iron anode and the solution due to the solution pressure of the metal diminished. On the contrary, the potential difference between the iron cathode and the solution due to the solution pressure of the metal was strengthened. Consequently the effect of the stirring of the electrolyte near the electrode was more conspicuous at the cathode than at the anode, in the case of a strong current. With a weak current, however, as the external electromotive force applied to the electrodes of the cell is small, the stirring effect naturally appears on the same scale at both electrodes.

As is well known in physiology, when one part of the tissue is put in action it becomes electronegative with respect to the other resting part. The cause of this phenomenon may be ascribed to some electrochemical changes and other more complicated ones; but the disturbance of the double layer, formed between a cell or other living particle and the body juice, in consequence of a stirring of the latter may, perhaps, also play a part in it.

The writer's cordial thanks are due to Prof. Mizuno and Lecturer Yoshida for their kind advice.

The Effect of an Electric Field on the Spectrum Lines of Hydrogen.

By

Shigeharu Nitta.

(Received July 17, 1917.)

Quite recently, adopting Lo Surdo's method, T. Takamine and U. Yoshida investigated the effect of an electric field on the spectrum lines of hydrogen¹ and helium².

With the same experimental arrangement, the writer examined the mode of decomposition of the Balmer lines $H\beta$, $H\delta$, and $H\epsilon$, as well as that of a certain number of lines belonging to the secondary spectrum of hydrogen. A seven-prism spectrograph and a two-prism spectrograph were used in the experiment.

The seven-prism spectrograph was constructed of four large 60° prisms and three large 45° prisms. The spectrum was photographed with a portrait lens having a relative aperture of 3.5 and a focal length of 18 cm. The dispersion on the photographic plate was 9.5 Å.U. per mm. at $H\beta$. This instrument was adopted for this line and also for the lines belonging to the secondary spectrum of hydrogen lying in the neighbourhood of $H\beta$.

The two-prism spectrograph was constructed with two large 60° prisms. The spectrum was photographed with a Zeiss Tessar having a relative aperture of 4.5 and a focal length of 18 cm. The dispersion on the photographic plate was 22.6 Å.U. per mm. at $H\alpha$.

About 30 photographs were taken with Ilford special rapid plates, the exposure being from 50 minutes to 2 hours. The primary current of the induction coil was about 3 amperes, while the secondary ranged from 0.3 to 0.8 milliamperes.

¹ T. Takamine and U. Yoshida, Mem. Col. of Sci., Kyoto, **2**, 137, 321 (1917).

² T. Takamine and U. Yoshida, Mem. Col. of Sci., Kyoto, **2**, 325 (1917).

To determine the field intensity, the following data for the outer components of H_γ given by Stark were relied on:—

4.6 Å.U. per 10^4 volt/cm. for the parallel (p) components.

3.2 Å.U. per 10^4 volt/cm. for the perpendicular (s) components.

At several different points, corresponding to various distances from the cathode, the amount of separation of the outer p- and s-components of H_γ was measured on the photographic plate obtained by the two-prism spectrograph.

In the present experiments, after the side tube leading to Kipp's apparatus was sealed, the evacuation was carried on until the end of Crookes dark space became very indistinct. Then the side tube leading to Gaede pump was sealed. In this manner a field intensity far greater than those employed by the former investigators was attained. Assuming the proportionality between the separation of the outer components of H_γ and the field intensity, the greatest field strength thus attained in the present experiment was 14.6×10^4 volt/cm. at the immediate neighbourhood of the cathode. The decomposition of H_β , in this case photographed by the seven-prism spectrograph, is reproduced in Fig. 2, Pl. I.

(A) H_β .

For the line H_β , Stark found 11 p- and 11 s-components in the case of "Feinzerlegung." With Lo Surdo's method, however, only 2 p- and 2 s-components have been hitherto obtained by various investigators.

In the present experiment, 6 s-components were clearly observed; and, though far more indistinct than these s-components, 6 p-components were observed beside the undisturbed line. The photograph showing the finer decomposition of the H_β line is reproduced in Fig. 1, Pl. I.

For the s-components of H_β , the amount of separation of the components was found to be approximately symmetrical with respect to the central line. Further, as shown in Table I, the separations were proportional to the field intensity E.

TABLE I.

Components E in 10 ⁴ volt/cm.	+3 (Intensity 5)		+2 (Intensity 10)		+1 (Intensity 3)		-1 (Intensity 3)		-2 (Intensity 10)		-3 (Intensity 5)	
	$\delta\lambda$ in Å.U.	$\delta\lambda/E$	$\delta\lambda$ in Å.U.	$\delta\lambda/E$	$\delta\lambda$ in Å.U.	$\delta\lambda/E$	$\delta\lambda$ in Å.U.	$\delta\lambda/E$	$\delta\lambda$ in Å.U.	$\delta\lambda/E$	$\delta\lambda$ in Å.U.	$\delta\lambda/E$
8.45	+8.61	1.02	+6.32	0.75	+2.79	0.33	-2.91	0.34	-6.01	0.71	-8.45	1.00
6.40	+6.20	0.97	+4.70	0.73	+2.21	0.34	-2.22	0.35	-4.61	0.72	-6.35	0.99
4.73	+4.70	0.99	+3.60	0.76	+1.70	0.36	-1.51	0.32	-3.52	0.74	-4.73	1.00
3.20	+3.41	1.06	+2.40	0.75	+1.14	0.36	-1.04	0.32	-2.20	0.69	-3.21	1.00
2.20	+2.20	1.00	+1.65	0.75	+0.86	0.39	-0.75	0.34	-1.54	0.70	-2.21	1.00
1.38	+1.42	1.03	+0.94	0.68	+0.42	0.30	-0.44	0.32	-0.95	0.69	-1.41	1.02
Mean of $\delta\lambda/E$		1.01		0.74		0.35		0.33		0.71		1.00

Similar measurements were performed on the plate reproduced in Fig. 2; and it was found that, for the intense components (+3), (+2), (-3), and (-2), the proportionality held good up to the field of 14.6×10^4 volt/cm. The results of these measurements are tabulated in Table 2.

TABLE 2.

F in 10 ⁴ volt/cm.		14.6	13.2	11.9	10.8	9.8	8.8	7.6	6.0	5.4	Mean of $\delta\lambda/E$
+3	$\delta\lambda$ in Å.U.	+15.5	+13.5	+12.2	+11.3	+10.3	+9.4	+7.4	+6.1	+5.3	1.03
	$\delta\lambda/E$	1.06	1.02	1.02	1.04	1.05	1.06	.97	1.02	.98	
+2	$\delta\lambda$ in Å.U.	+11.0	+9.8	+8.6	+8.0	+7.3	+6.5	+5.9	+4.2	+3.7	0.74
	$\delta\lambda/E$	.76	.74	.72	.74	.74	.74	.78	.70	.69	
-2	$\delta\lambda$ in Å.U.	-10.7	-10.0	-8.4	-7.7	-7.1	-6.5	-5.8	-4.0	-3.6	0.72
	$\delta\lambda/E$	.73	.76	.71	.72	.72	.74	.76	.67	.67	
-3	$\delta\lambda$ in Å.U.	-16.0	-13.0	-12.3	-11.4	-10.4	-9.5	-7.5	-6.2	-5.4	1.03
	$\delta\lambda/E$	1.10	.99	1.04	1.06	1.06	1.08	.99	1.04	1.00	

As shown in Table 3, the 6 components correspond to the components (+3), (+2), (+1), (-1), (-2), and (-3) given by Stark in the case of "Feinzerlegung" of H β .

TABLE 3.

The s-components of H_β at $E = 7.4 \times 10^4$ volt/cm.

		+3	+2	+1	-1	-2	-3
Stark	$\delta\lambda$ in Å.U.	+7.3	+5.8	+2.5	-2.5	-5.0	-7.4
Nitta	$\delta\lambda$ in Å.U. (calc.)	+7.5	+5.4	+2.6	-2.4	-5.3	-7.5

In this case, values of E were determined by the p- and s-components of H_γ photographed simultaneously by the two-prism spectrograph.

For the p-components, it was found that the six components corresponded to the components (+5), (+4), (+3), (-3), (-4), and (-5) given by Stark in the case of "Feinzerlegung" of H_β .

(B) H_δ .

For H_δ four s-components and two p-components were observed, as shown in Fig. 3, Pl. 1.

The following tables (Table 4 and Table 5) give the relation between the field intensity and the amount of displacement $\delta\lambda$ from the position of the undisturbed line.

TABLE 4.

The p-components of H_δ .

E in 10^4 volt/cm.						Mean of $\delta\lambda/E$
Components		8.4	4.4	2.4	1.2	
+1	$\delta\lambda$ in Å.U.	+25.5	+14.5	+8.0	+4.0	3.25
(Intensity) 2	$\delta\lambda/E$	3.03	3.29	3.33	3.33	
-1	$\delta\lambda$ in Å.U.	-25.0	-15.4	-8.3	-4.0	3.31
(Intensity) 2	$\delta\lambda/E$	2.98	3.50	3.46	3.33	

TABLE 5.
The s-components of H_{δ} .

E in 10^4 volt/cm.		8.4	5.2	3.9	1.5	Mean of $\delta\lambda/E$
Components						
+3 (Int. 2)	$\delta\lambda$ in Å.U.	20.0	12.4	8.6	4.1	2.42
	$\delta\lambda/E$	2.38	2.38	2.20	2.73	
+1 (Int. 5)	$\delta\lambda$ in Å.U.	8.1	4.2	2.9	1.4	0.86
	$\delta\lambda/E$	0.96	0.81	0.74	0.93	
-1 (Int. 5)	$\delta\lambda$ in Å.U.	7.7	4.1	3.0	1.1	0.80
	$\delta\lambda/E$	0.92	0.79	0.77	0.73	
-2 (Int. 2)	$\delta\lambda$ in Å.U.	21.0	13.0	9.0	4.1	2.51
	$\delta\lambda/E$	2.50	2.50	2.31	2.73	

Thus the separation is approximately symmetrical and is proportional to the field intensity.

As shown in the following table (Table 6), the p- and s-components were all well identified with those of Stark given in the case of "Grobzerlegung" of H_{δ} .

TABLE 6.
Components of H_{δ} at $E=2.85 \times 10^4$ volt/cm.

		p-components		s-components			
		+1	-1	+2	+1	-1	-2
Stark	$\delta\lambda$ in Å.U.	8.54	9.16	6.27	1.96	2.36	6.72
Nitta	$\delta\lambda$ in Å.U. (calc.)	9.1	9.4	6.9	2.4	2.3	7.1

(C) H_{δ} .

For this line, Stark gives only two p-components. Adopting Lo Surdo's method, Sonaglia¹ found two p- and five s-components, but he did not measure the amount of separation.

The result obtained in the present experiment was by no means satisfactory. Fairly good images of the two p-components were obtained in a few cases; while, for the s-components, two outer and two inner components appeared very indistinctly.

¹ Sonaglia, Rendiconti d. Lincei, **24**, 621, (1916).

The numerical values obtained are tabulated below (Table 7, 8, and 9).

TABLE 7.

The p-components of H_{ϵ} .

E in 10^4 volt/cm.		6.9	4.8	3.25	Mean of $\delta\lambda/E$.
+1	$\delta\lambda$ in Å.U.	+30.0	+19.8	+14.2	4.28
	$\delta\lambda/E$	4.35	4.13	4.37	
-1	$\delta\lambda$ in Å.U.	-27.5	-18.8	-12.4	3.91
	$\delta\lambda/E$	4.00	3.92	3.81	

TABLE 8.

The s-components of H_{ϵ} .

E in 10^4 volt/cm.	6.9	3.5	0.9	Mean of $\delta\lambda/E$
Distance between the outer components in Å.U.	35.0	17.5	5.7	5.5
$\delta\lambda/E$	5.1	5.0	6.3	
Distance between the inner components in Å.U.	6.3			
$\delta\lambda/E$	0.91			

TABLE 9.

$\delta\lambda$ in Å.U. at $E = 2.85 \times 10^4$ volt/cm.

	+1	-1
Stark	+12.2	-11.3
Nitta (calc.)	+12.1	-11.1

(D) Secondary spectrum lines of hydrogen.

In the present experiment, the regions in which a number of affected lines were measured, were from $\lambda 4340$ to $\lambda 3900$ and in the neighbourhood of H_{β} .

The wave-lengths of the lines in these regions, which were determined by the comparison spectrum of the iron arc, were found to agree well with those obtained by Watson and given in Kayser's *Handbuch der Spectroscopie*. In the following, the writer has adopted Watson's values to the first decimal of an Å.U.

In Fig. 4, the curves showing various modes of decomposition presented by the affected lines have been roughly drawn. With the dispersion used in the present experiment, it was difficult to measure the amounts of displacement in the region above the dotted line.

Summary.

1. Employing Lo Surdo's method, the effect of an electric field on the spectrum lines of hydrogen has been investigated.
2. To determine the field intensity, the data given by Stark for H_γ in the case of "Grobzerlegung" were relied on.
3. By lowering the pressure in the discharge tube until the end of Crookes dark space became very indistinct, a field intensity far greater than those employed by the former investigators has been attained, its maximum value being about 14.6×10^4 volt/cm.
4. For H_β 6 s-components have been observed and identified with those obtained by Stark. The proportionality between the amount of separation and the field intensity was found to hold good for these components.
5. For H_δ , 2 p- and 4 s-components, and for H_ϵ 2 p- and 4 s-components have been obtained; and excepting the 4 s-components of H_ϵ , identified with the components obtained by Stark.
6. The influence of the electric field on certain lines belonging to the secondary spectrum lines of hydrogen has been investigated.

In conclusion, the writer wishes to express his sincere thanks to Assist. Prof. T. Takamine and Lecturer U. Yoshida for their kind guidance in the course of the experiment. The writer's thanks are also due to Prof. M. Fukui of Kyoto Higher Technical School for the loan of a Krüss spectrometer.

Fig. 1.

Exposure : 2h

$E_{\max.}$: 11.6×10^4 volt/cm.

Magnification : 8

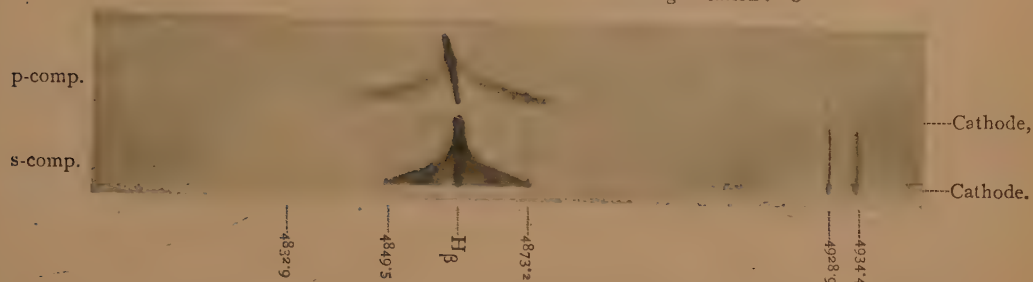


Fig. 2.

Exposure : 50m

$E_{\max.}$: 14.6×10^4 volt/cm.

Magnification : 8



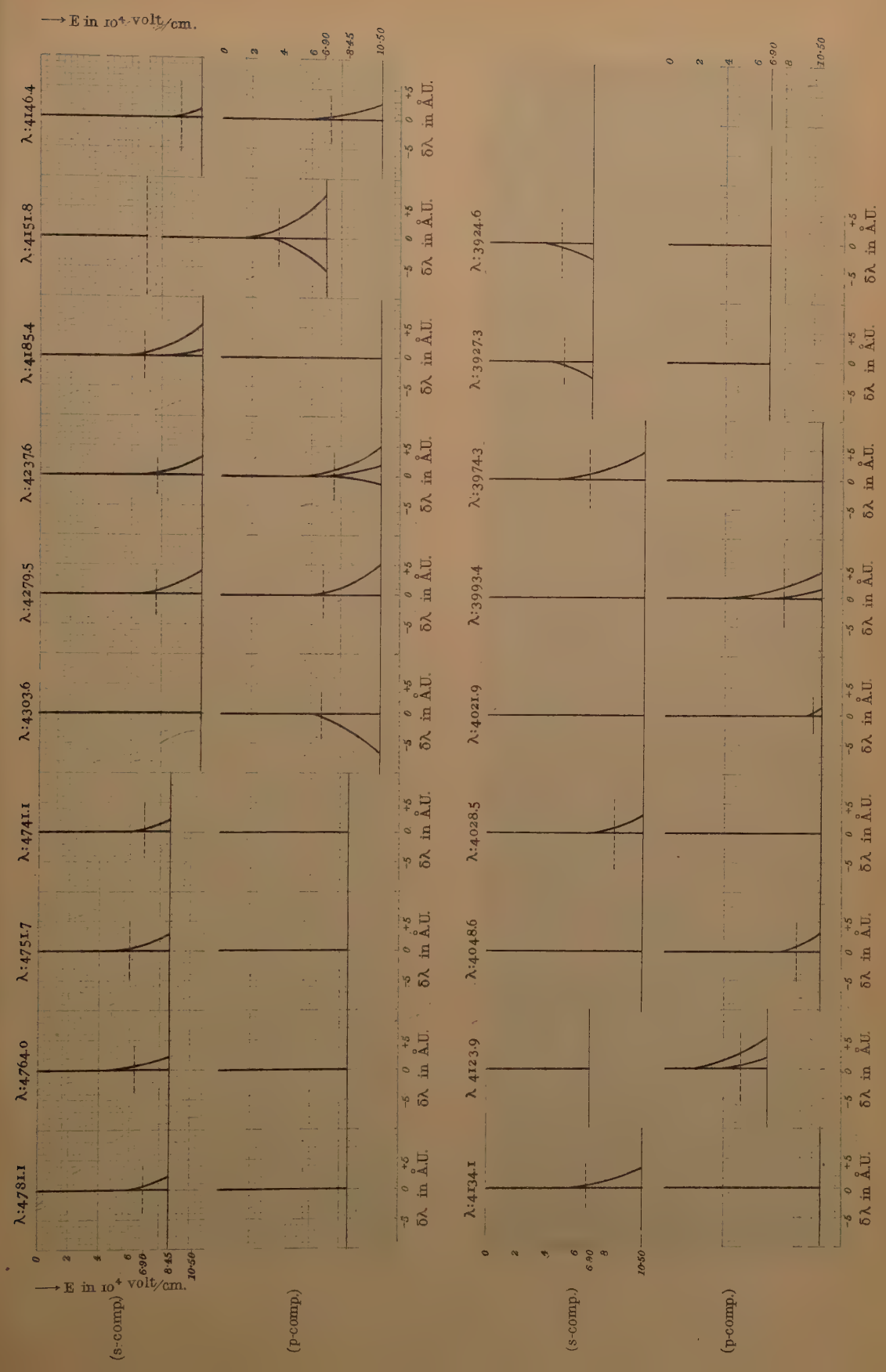
Fig. 3.

Exposure : 1h 40m

$E_{\max.}$: 8.45×10^4 volt/cm.

Magnification : 5





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